

Algebraic Surfaces in Positive Characteristic

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ABSTRACT. These notes are intended as an overview and an introduction to the theory of algebraic surfaces in positive characteristic. We assume that the reader has a background in complex geometry and has seen the Kodaira–Enriques classification of complex surfaces before.

Introduction

These notes are meant for complex geometers, who would like to see geometry over fields of positive characteristic. More precisely, these notes are about the theory of algebraic surfaces over such fields.

Very roughly speaking, the theory of curves in characteristic zero and positive characteristic look pretty similar, although there are, of course, many differences. For example, curves lift from characteristic p to characteristic zero. Also, the theory of curves was already studied in positive characteristic from a very early stage on.

However, from dimension two on, geometry over fields of positive characteristic has long been considered "pathological" as even in the title of [Mu61] or "exotic". Thus, the emphasis was more on finding differences rather than observing that geometry in positive characteristic is often not so different from characteristic zero when looking from the right angle.

In their three fundamental articles [Mu69], [B-M2] and [B-M3], Mumford and Bombieri established the Kodaira–Enriques classification over fields of positive characteristic. Together with Artin's results [Ar62] and [Ar66] on singularities, especially rational singularities and Du Val singularities, as well as work of Ekedahl [Ek88] on pluricanonical systems of surfaces of general type this sets the scene for surfaces over fields of positive characteristic.

As over the complex numbers, there is a vast number of examples and (partial) classification results of special classes of surfaces over fields of positive characteristic. It is impossible to mention all of them, but it should be remarked that the Japanese contributed a lot.

In particular, my notes reflect only a small amount what could be said in this field. This means that I had to leave out many important topics and may have given too much space to topics that particularly interest myself. In the first part of these notes I deal with the Kodaira–Enriques classification. More precisely, I cover

- a little bit on the Frobenius morphism(s), curves and group schemes,
- a little bit about Betti- and Hodge numbers,
- the structure of birational maps and minimal models, which turns out to be pretty much the same as in characteristic zero,
- elliptic fibrations and the new phenomenon of quasi-elliptic fibrations,
- the Kodaira–Enriques classification,
- a detailed classification of surfaces of Kodaira dimension zero with a special emphasis on the new classes, and then

- pluricanonical maps for surfaces of general type and classical inequalities for surfaces of general type.

The remaining sections deal with more specialized topics. These sections are independent from each other and from the Kodaira–Enriques classification in the sections before. More precisely, I discuss

- that unirational surfaces can have arbitrary Kodaira dimension and the notions of supersingularity due to Artin and Shioda,
- a little bit about lifting to characteristic zero,
- rational and Du Val singularities, which turn out to behave very similar to characteristic zero, and finally
- inseparable morphisms and their description via foliations.

I should mention that the reader who is interested in learning surface theory over arbitrary fields from scratch (including proofs) is advised to read Badescu’s excellent book [Ba]. From there the reader can continue with more advanced topics including the original articles by Bombieri and Mumford mentioned above, and all the literature given in these notes.

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1. Frobenius, curves and groups

Before dealing with surfaces, we first shortly review a little bit of background material. Of course, the omnipresent Frobenius morphism has to be mentioned first – in many cases, when a characteristic zero argument breaks down in positive characteristic, Frobenius is responsible. On the other hand, this special morphism also provides in many situations the solution to a problem. Next, we give a very short overview over curves and over group actions. We have chosen our material in view of what we need for the classification and description of surfaces later on.

Frobenius. Let us recall that a field k of positive characteristic p is *perfect* if and only if its Frobenius morphism $x \mapsto x^p$ is surjective, i.e., if every element in k has a p -th root in k . For example, finite fields and algebraically closed fields are perfect.

Let X be an n -dimensional variety over a field k with structure morphism $f : X \rightarrow \operatorname{Spec} k$. Then, the morphism that is the identity on points of X and is $x \mapsto x^p$ on the structure sheaf \mathcal{O}_X is called the *absolute Frobenius morphism* F_{abs} of X .

However, the absolute Frobenius morphism is not “geometric” in the sense that its effect is also $x \mapsto x^p$ on the ground field and thus non-trivial, except for $k = \mathbb{F}_p$. To obtain a morphism over k , we first form the pull-back

$$X^{(p)} := X \times_{\operatorname{Spec} k} \operatorname{Spec} k \xrightarrow{\operatorname{pr}_2} \operatorname{Spec} k$$

with respect to the structure map $f : X \rightarrow \operatorname{Spec} k$ and with respect to the absolute Frobenius $F_{\text{abs}} : \operatorname{Spec} k \rightarrow \operatorname{Spec} k$. This *Frobenius pullback* $f^{(p)} := \operatorname{pr}_2 : X^{(p)} \rightarrow \operatorname{Spec} k$ is a new variety over k . If k is perfect (see below) then X and $X^{(p)}$ are abstractly isomorphic as schemes, but not as varieties over k .

Now, by the universal property of pull-backs we obtain a morphism of varieties over k , the *k -linear Frobenius morphism* $F : X \rightarrow X^{(p)}$

$$\begin{array}{ccccc}
 X & & & & \\
 \swarrow F & & \searrow F_{\text{abs}} & & \\
 & X^{(p)} & \xrightarrow{\quad} & X & \\
 \searrow f & \downarrow f^{(p)} & & \downarrow f & \\
 & \operatorname{Spec} k & \xrightarrow{F_{\text{abs}}} & \operatorname{Spec} k &
 \end{array}$$

In more down to earth terms and for affine space this simply means

$$\begin{array}{llll}
 & k[x_1, \dots, x_n] & \rightarrow & k[x_1, \dots, x_n] \\
 \text{absolute} & : & f(x_1, \dots, x_n) & \mapsto (f(x_1, \dots, x_n))^p \\
 k\text{-linear} & : & c & \mapsto c \\
 & & x_i & \mapsto x_i^p
 \end{array} \quad \text{if } c \in k$$

In the geometry over finite fields there are yet more Frobenius morphisms: over a field \mathbb{F}_q with $q = p^n$ elements one has a Frobenius morphism $x \mapsto x^q$, and for technical reasons sometimes its inverse has to be considered. Depending on author

and context all these morphisms and various base-changes are called “Frobenius” and so a little care is needed.

Now, if X is n -dimensional over k , then $F : X \rightarrow X^{(p)}$ is a morphism of degree p^n . Moreover, if k is perfect then, on the level of function fields, this morphism corresponds to the inclusion

$$k(X^{(p)}) = k(X)^p \subseteq k(X).$$

Note that $k(X)^p$, the set of p .th powers of $k(X)$ is in fact a field: it is not only closed under multiplication but also under addition since $x^p + y^p = (x + y)^p$ in characteristic p . Let us also fix an algebraic closure \overline{K} of $K = k(X)$. For every integer $i \geq 0$ we define

$$K^{p^{-i}} := \{x \in \overline{K} \mid x^{p^i} \in K\}$$

and note that these sets are in fact fields. Moreover, the field $K^{p^{-i}}$ is a finite and purely inseparable extension of K of degree p^{ni} . The limit $K^{p^{-\infty}}$ as i tends to infinity is called the *perfect closure* of K in \overline{K} , as it is the smallest subfield of \overline{K} that is perfect and that contains K .

Definition 1.1. Let L be a finite and purely inseparable extension of K . The *height* of L over K is defined to be the minimal i such that $K \subseteq L \subseteq K^{p^{-i}}$.

Similarly, if $\varphi : Y \rightarrow X$ is a purely inseparable and generically finite morphism of varieties over a perfect field then *height of φ* is defined to be the height of the extension of function fields $k(Y)/k(X)$. For example, the k -linear Frobenius morphism is of height one.

For more on inseparable morphisms, we refer to Section 11 and the references there.

Curves. Most of the results of this section are classical, and we refer to [Hart, Chapter IV] or [Liu02] for details, specialized topics and further references. Let C be a smooth projective curve over an algebraically closed field of characteristic $p \geq 0$. Then its *geometric genus* is defined to be

$$g(C) = h^0(C, \omega_C) = h^1(C, \mathcal{O}_C),$$

where ω_C denotes the dualizing sheaf. The first equality is the definition, and the second equality follows from Serre duality. Since C is smooth over k , ω_C is isomorphic to the sheaf of Kähler differentials $\Omega_{C/k}$.

Let $\varphi : C \rightarrow D$ be a finite morphism between curves. Then the *Riemann–Hurwitz formula* states that there is a linear equivalence of divisors on C

$$K_C \sim \varphi^*(K_D) + \sum_{P \in C} \text{length}(\Omega_{C/D})_P \cdot P.$$

Here, $\Omega_{C/D}$ is the sheaf of relative differentials. In case φ is a separable morphism, i.e., generically étale, this is a torsion sheaf supported at the points where φ is not étale, i.e., the ramification points. We recall that the ramification at a ramification

point P is called *tame*, if the ramification index e_P at P is not divisible by $p = \text{char}(k)$, and that it is called *wild* otherwise. We have

$$\text{length}(\Omega_{C/D})_P \begin{cases} = & e_P - 1 & \text{if } P \text{ is tame} \\ > & e_P - 1 & \text{if } P \text{ is wild} \end{cases} .$$

An important case where one can say more about wild ramification is in case φ is a Galois morphism: then one can define for every wild ramification point certain subgroups of the Galois group, the so-called *higher ramification groups*, that control the length of $\Omega_{C/D}$ at P , cf. [Se68, Chapitre IV.1]

Galois covers with group $\mathbb{Z}/p\mathbb{Z}$ are called *Artin–Schreier covers* and an example is $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by the affine equation

$$z^p - z = t .$$

Note that it is branched only over $t = \infty$. In particular, φ yields a non-trivial étale cover of \mathbb{A}^1 , which implies that the affine line \mathbb{A}^1 is *not* algebraically simply connected. However, it is still true that every étale cover of \mathbb{P}^1 is trivial, i.e., \mathbb{P}^1 is algebraically simply connected.

There exists a sort of Riemann–Hurwitz formula in all dimensions in case φ is inseparable. We refer to [Ek87] for more information on $\Omega_{C/D}$ in this case. For example, the ramification divisor is defined only up to linear equivalence. On the other hand, the structure of purely inseparable morphisms for curves is simple: namely every such morphism is just the composite of k -linear Frobenius morphisms. However, from dimension two on, inseparable morphisms become more complicated. We will come back to these issues in Section 11.

If a curve has genus zero, then ω_C^\vee is very ample and embeds it as a quadric in \mathbb{P}_k^2 . Moreover, a quadric with a k -rational point is isomorphic to \mathbb{P}_k^1 over any field. Moreover, the Riemann–Hurwitz theorem implies that every curve that is dominated by a curve of genus zero also has genus zero (Lüroth’s theorem). Thus,

Theorem 1.2. *If $g(C) = 0$ then $C \cong \mathbb{P}_k^1$, i.e., C is rational. Moreover, every unirational curve, i.e., which is dominated by \mathbb{P}_k^1 , is rational.*

Although unirational surfaces are rational in characteristic zero, this turns out to be false in positive characteristic as we shall see in Section 8.

Theorem 1.3. *If $g(C) = 1$ then*

- (1) *after choosing a point $O \in C$ there exists the structure of an Abelian group on the points of C , i.e., C is an Abelian variety of dimension one.*
- (2) *The linear system $|2O|$ defines a finite morphism of degree 2*

$$\varphi : C \rightarrow \mathbb{P}^1 .$$

- (3) *There exists a j -invariant $j(C) \in k$ such that two genus one curves are isomorphic if and only if they have the same j -invariant.*
- (4) *If $p \neq 2$ then φ is branched over four points and j can be computed from the cross ratio of these four points.*

- (5) The linear system $|3O|$ embeds C as a cubic curve into \mathbb{P}_k^2 . Moreover, if $p \neq 2, 3$ then C can be given by an affine equation

$$y^2 = x^3 + ax + b$$

for some $a, b \in k$.

We note that the description of elliptic curves as quotients of \mathbb{C} by lattices also has an analog in positive characteristic. This leads to the theory of *Drinfel'd modules*, which is parallel to the theory of elliptic curves in positive characteristic but has not so much to do with the theory of curves, see [Go96, Chapter 4].

We recall that a curve C of genus $g \geq 2$ is called *hyperelliptic* if there exists a separable morphism φ of degree 2 from C onto \mathbb{P}^1 . If $p = \text{char}(k) \neq 2$ then φ is branched over $2g + 2$ points. On the other hand, if $p = 2$ then every ramification point is wildly ramified and thus there at most $g + 1$ branch points in this case. In any case, all curves of genus $g = 2$ are hyperelliptic and the generic curve of genus $g \geq 3$ is not hyperelliptic.

Theorem 1.4. *If $g(C) \geq 2$ then $\omega_C^{\otimes 2}$ is very ample if and only if C is not hyperelliptic. In any case, $\omega_C^{\otimes 3}$ is very ample for all curves with $g(C) \geq 3$ and $\omega_C^{\otimes 4}$ is very ample for all curves of general type.*

Deligne and Mumford have shown there exists a Deligne–Mumford stack, flat over $\text{Spec } \mathbb{Z}$ and of dimension $3g - 3$ classifying curves of genus $g \geq 2$. Thus, the moduli spaces in various characteristics arise by reducing the one over $\text{Spec } \mathbb{Z}$ modulo p .

Let us finally mention a couple of facts concerning automorphism groups:

- (1) If $p \neq 2, 3$ then the automorphism group of an elliptic curve, i.e., automorphisms fixing the neutral element O , has order 2, 4 or 6.
- (2) However, the elliptic curve with $j = 0$ has 12 automorphisms if $p = 3$ and even 24 automorphisms if $p = 2$, see [Sil86, Theorem III.10.1].
- (3) The automorphism group of a curve of genus $g \geq 2$ is finite. However, Hurwitz's bound $84(g - 1)$ on automorphisms in characteristic zero may be violated. See [Hart, Chapter IV.2, Exercise 2.5] for details and further references

This will later also be important for the classification of surfaces. For example, hyperelliptic surfaces arise as quotients $(E_1 \times E_2)/G$, where E_1, E_2 are elliptic curves with G -actions - thus, new classes in characteristic 2, 3 are also related to elliptic curves having larger automorphism groups in these characteristics.

Groups. Constructions with groups are ubiquitous in geometry. Instead of finite groups we will consider *finite and flat group schemes* G over a ground field k , which we assume to be algebraically closed of characteristic $p \geq 0$. Thus, $G = \text{Spec } A$ for some finite-dimensional k -algebra A , and we assume that there exist morphisms

$$O : \text{Spec } k \rightarrow G \quad \text{and} \quad m : G \times G \rightarrow G$$

where m denotes multiplication, and O the neutral element. We refer to [Wa79, Chapter I] for the precise definition and note that it amounts to say that A is a Hopf-algebra. The dimension $\dim_k A$ is called the *length* of the group scheme G .

Let us recall the following construction: for a finite group of order n with elements g_1, \dots, g_n one obtains a finite flat group scheme by taking the union of n disjoint copies of $\text{Spec } k$, one representing each g_i , and defining m via the multiplication in the group one started with. This is the *constant group scheme* associated to a finite group.

Whenever the length $\ell(G)$ is coprime to $p = \text{char}(k)$ then G is constant. In particular, in characteristic zero one obtains an equivalence between the categories of finite groups and finite flat group schemes (note that the ground field is assumed to be algebraically closed).

Now, the constant group scheme $\mathbb{Z}/p\mathbb{Z}$ is an example of a group scheme of length p . As an algebra, it is isomorphic to k^p and thus étale over k . On the other hand, on $\text{Spec } k[x]/(x^p)$ one can define even two structures of group schemes - these are examples of *infinitesimal group schemes*. Let us first note the following theorem, which tells us that infinitesimal group schemes are a particular characteristic p phenomenon:

Theorem 1.5 (Cartier). *Group schemes over fields of characteristic zero are smooth and thus reduced.*

Denote by \mathbb{G}_a the group scheme corresponding to the additive group and by \mathbb{G}_m the group scheme corresponding to the multiplicative group of k , see [Wa79, Chapter I.2] - these group schemes are affine but not finite over k . Then the first example of an infinitesimal group scheme is μ_p , the subgroup scheme of p -th roots of unity. Thus there exists a short exact sequence of group schemes (in the flat topology)

$$0 \rightarrow \mu_p \rightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^p} \mathbb{G}_m \rightarrow 0$$

Note that its infinitesimality is caused by the fact that $x^p - 1 = (x - 1)^p$ in characteristic p . The second example is α_p , the kernel of Frobenius on \mathbb{G}_a , i.e., we have a short exact sequence

$$0 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \rightarrow 0$$

Both group schemes, α_p and μ_p , are isomorphic to $\text{Spec } k[x]/(x^p)$ as schemes, and are thus infinitesimal (non-reduced), but have different multiplication maps.

Theorem 1.6. *Over an algebraically closed field of characteristic p there are three finite and flat group schemes of length p , namely $\mathbb{Z}/p\mathbb{Z}$, μ_p and α_p .*

For more general results we refer to [O-T70].

For example, if E is an elliptic curve in characteristic p then multiplication by p induces a morphism $E \rightarrow E$, whose kernel $E[p]$ is a finite flat group scheme of length p^2 (as one would expect from characteristic zero), and

$$E[p] \cong \begin{cases} \text{either} & \mu_p \oplus (\mathbb{Z}/p\mathbb{Z}) & \text{and } E \text{ is called } \textit{ordinary} \\ \text{or} & M_2 & \text{a non-split extension of } \alpha_p \text{ by itself} \\ & & \text{and } E \text{ is called } \textit{supersingular} \end{cases}$$

As the name suggests, the generic elliptic curve is ordinary. In fact, by a theorem of Deuring, there are approximately $p/12$ supersingular elliptic curves in characteristic p , see [Sil86, Theorem V.4.1].

When constructing algebraic varieties, one often constructs them via covers of known varieties, and then Galois covers are particularly convenient. In characteristic p , the way to construct a “pathological” example is often via constructing a variety as a purely inseparable cover of a known one, and the role of Galois covers is played by torsors under these infinitesimal group schemes α_p and μ_p . We come back to these things in Section 11.

2. Cohomological tools and invariants

The results of this section circle around the purely algebraic versions of Betti numbers, Hodge numbers, and deRham-cohomology. Especially towards the end, the subjects get deeper, our exposition becomes sketchier and we advise the reader interested in surface theory only, to skip all but the first three paragraphs.

For the whole section X will be a smooth and projective variety of arbitrary dimension over an algebraically closed field of characteristic $p \geq 0$.

Hodge numbers. We define the *Hodge numbers* of X to be

$$h^{i,j}(X) = \dim_k H^j(X, \Omega_X^i).$$

We note that Serre duality holds for projective Cohen–Macaulay schemes over any field [Hart, Chapter III.7], and in particular we obtain

$$h^{i,j}(X) = h^{n-i, n-j}(X), \quad \text{where } n = \dim(X).$$

However, even for a smooth projective surface the numbers $h^{0,1}(S)$ and $h^{1,0}(S)$ may be different, since this equality over the complex numbers comes from complex conjugation. We refer to [Li08, Theorem 8.3] for examples of surfaces in characteristic 2 where $h^{1,0} - h^{0,1}$ gets arbitrarily large.

Betti numbers. One algebraic replacement for singular cohomology is ℓ -adic cohomology, whose construction is due to Grothendieck. We refer to [Hart, Appendix C] for motivation as well as to [Mil] for a complete treatment. Let us here only describe its properties: let ℓ be a prime number with $\ell \neq p$ and let \mathbb{Q}_ℓ be the field of ℓ -adic numbers, i.e., the completion of \mathbb{Q} with respect to the ℓ -adic valuation. Then

- (1) the ℓ -adic cohomology groups $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$ are finite-dimensional \mathbb{Q}_ℓ -vector spaces,
- (2) they are zero for $i < 0$ and $i > 2 \dim(X)$,
- (3) the dimension of $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$ is independent of ℓ (here, $\ell \neq p$ is crucial), and is denoted by $b_i(X)$, the i .th *Betti number*,
- (4) $H_{\text{ét}}^*(X, \mathbb{Q}_\ell)$ satisfies Poincaré duality.

If $k = \mathbb{C}$, then so-called *comparison theorems* show that these Betti numbers coincide with the topological ones. Let us also mention the following feature: if k is not algebraically closed then the absolute Galois group $\text{Gal}(\bar{k}/k)$ acts on ℓ -adic

cohomology and may give rise to interesting representations of the absolute Galois group.

Let k be an arbitrary field. Then, for the following two classes of varieties, ℓ -adic cohomology and Hodge invariants are precisely as one would expect them from complex geometry:

- (1) If C is a smooth curve over a field k , then $b_0 = b_2 = 1$, $b_1 = 2g$ and $h^{1,0} = h^{0,1} = g$.
- (2) If A is an Abelian variety of dimension g over a field k then $b_0 = b_{2g} = 1$, $b_1 = 2g$ and $h^{0,1} = h^{1,0} = g$. Moreover, there exists an isomorphism

$$\Lambda^i H_{\text{ét}}^1(A, \mathbb{Q}_\ell) \cong H_{\text{ét}}^i(A, \mathbb{Q}_\ell)$$

giving – among many other things – the expected Betti numbers.

However, for more general classes of smooth and projective varieties and over fields of positive characteristic, the relations between Betti numbers, Hodge invariants, deRham-cohomology and the Frölicher spectral are more subtle than over the complex numbers, as we shall see below.

Let us first discuss $h^{1,0}$, $h^{0,1}$ and b_1 in more detail, since this is important for the classification of surfaces. Also, these numbers can be treated fairly elementary.

Picard scheme and Albanese variety. If X is smooth and proper over a field k , then there exists an Abelian variety $\text{Alb}(X)$ over k , the *Albanese variety* of X , and a morphism

$$\text{alb}_X : X \rightarrow \text{Alb}(X)$$

the Albanese-morphism, such that every morphism from X to an Abelian variety factors over alb_X .

Next, the Picard functor, classifying line bundles on X , is representable by a group scheme, the *Picard scheme* $\text{Pic}(X)$ with neutral element \mathcal{O}_X . We denote by $\text{Pic}^0(X)$ the identity component of $\text{Pic}(X)$. From deformation theory we get a natural isomorphism

$$T\text{Pic}^0(X) \cong H^1(X, \mathcal{O}_X),$$

where $T\text{Pic}^0(X)$ denotes the Zariski-tangent space of $\text{Pic}^0(X)$ at the origin.

Now, group schemes over fields of positive characteristic may be non-reduced but one can show that the reduction of $\text{Pic}^0(X)$ is an Abelian variety, which is the dual Abelian variety of $\text{Alb}(X)$. Also, one can show that b_1 is twice the dimension of $\text{Alb}(X)$. Thus, we get

$$\frac{1}{2}b_1(X) = \dim \text{Alb}(X) = \dim \text{Pic}^0(X).$$

Moreover, the dimension of the Zariski tangent space to $\text{Pic}^0(X)$ is at least equal to the dimension of $\text{Pic}^0(X)$ and thus

$$h^{0,1}(X) = h^1(X, \mathcal{O}_X) \geq \frac{1}{2}b_1(X)$$

with equality if and only if $\text{Pic}^0(X)$ is a reduced group scheme, i.e., an Abelian variety. In Section 1 we already mentioned Cartier's theorem, according to which

group schemes over a field of characteristic zero are reduced. As a corollary, we obtain a purely algebraic proof of the following fact

Proposition 2.1. *A smooth and proper variety over a field of characteristic zero satisfies $b_1(X) = 2h^{0,1}(X)$.*

For curves and Abelian varieties over arbitrary fields the numbers b_1 , $h^{1,0}$ and $h^{0,1}$ are precisely as over the complex numbers. On the other hand, over fields of positive characteristic

- (1) there do exist surfaces with $h^{0,1} > b_1/2$, i.e., with non-reduced Picard schemes, see [Ig55b] and [Se58].

In [Mu66, Lecture 27], the non-reducedness of $\text{Pic}^0(X)$ is related to non-trivial Bockstein operations β_n from subspaces of $H^1(X, \mathcal{O}_X)$ to quotients of $H^2(X, \mathcal{O}_X)$. In particular, a smooth projective variety with $h^2(X, \mathcal{O}_X) = 0$ has a reduced $\text{Pic}^0(X)$, which applies, e.g., to curves. In the case of surfaces, a quantitative analysis of which classes can have non-reduced Picard schemes has been carried out in [Li09a].

Differential one-forms. We shall see in Section 8 that over fields of positive characteristic the pull-back of a non-trivial differential forms under a morphism may become zero. However, by a fundamental theorem of Igusa [Ig55a], every non-trivial global 1-form on $\text{Alb}(X)$ pulls back, via alb_X to a non-trivial global 1-form on X . We thus obtain the estimate

$$h^{1,0}(X) = h^0(X, \Omega_X^1) \geq \frac{1}{2}b_1(X)$$

Moreover, all global 1-forms coming from $\text{Alb}(X)$ are d -closed.

We have $h^{1,0} = b_1/2$ for curves and Abelian varieties over arbitrary fields and their global 1-forms are d -closed. On the other hand, over fields of positive characteristic

- (1) there do exist surfaces with $h^{1,0} > b_1/2$, i.e., with “too many” global 1-forms, see [Ig55b], and
- (2) there do exist surfaces with global 1-forms that are not d -closed [Mu61]. Such surfaces have non-trivial differentials in their Frölicher spectral sequence, which thus does not degenerate at E_1 .

We refer to [Ill79, Proposition II.5.16] for more results and to [Ill79, Section II.6.9] for the connection to Oda’s subspace in first deRham cohomology.

Igusa’s inequality. We denote by ρ the rank of the Néron–Severi group, which is always finite. More precisely, Igusa’s theorem [Ig60] states

$$\rho(X) \leq b_2(X).$$

Nowadays this can be explained by a Chern map from $\text{NS}(X)$ to second étale or crystalline cohomology.

On the other hand, one can define a cycle map $\text{NS}(X) \otimes_{\mathbb{Z}} k \rightarrow H^1(X, \Omega_X^1)$ purely algebraically [Ba, Exercise 5.5], but this map may fail to be injective, as

the supersingular Fermat surfaces in [Sh74] show. Moreover, these examples also violate the inequality $\rho \leq h^{1,1}$.

Kodaira vanishing. There are counter-examples by Raynaud [Ra78] to the Kodaira vanishing theorem for surfaces in positive characteristic. However, we mention the following results that tell us that the situation is not too bad:

- (1) if \mathcal{L} is an ample line bundle then $\mathcal{L}^{\otimes \nu}$, $\nu \gg 0$ fulfills Kodaira vanishing, [Hart, Theorem III.7.6].
- (2) if a variety lifts over W_2 then Kodaira vanishing holds [Ill02, Theorem 5.8], see also Section 9.
- (3) Kodaira vanishing holds for the (rather special) class of *Frobenius-split* varieties [B-K05, Theorem 1.2.9]
- (4) in [Ek88, Section II], Ekedahl develops tools to handle possible failures of Kodaira vanishing, see also Section 7.

Frölicher spectral sequence. Let Ω_X^i be the sheaf of (algebraic) differential i -forms. Then differentiating induces a complex, the (*algebraic*) *deRham-complex* (Ω_X^*, d) . Since the sheaves Ω_X^i are not acyclic with respect to the Zariski topology, we define the *deRham-cohomology* $H_{\text{dR}}^*(X/k)$ to be the hypercohomology of this complex. In particular, there is always a spectral sequence

$$E_1^{i,j} = H^j(X, \Omega_X^i) \Rightarrow H_{\text{dR}}^{i+j}(X/k),$$

the *Frölicher spectral sequence*, from Hodge- to deRham-cohomology. If $k = \mathbb{C}$, then these cohomology groups and the spectral sequence coincide with the analytic ones. Already the existence of such a spectral sequence implies for all $m \geq 0$ the inequality

$$\sum_{i+j=m} h^j(X, \Omega_X^i) \geq h_{\text{dR}}^m(X/k).$$

Equality for all m is equivalent to the degeneration of this spectral sequence at E_1 . Over the complex numbers, degeneration at E_1 is true but the classical proof uses methods from differential geometry, functional analysis and partial differential equations. On the other hand, if a variety of positive characteristic admits a lifting over W_2 (see Section 9), then there we have the following theorem [D-I87] (but see [Ill02] for an expanded version):

Theorem 2.2 (Deligne–Illusie). *Let X be a smooth and projective variety of dimension d in characteristic $p \geq d$ and assume that X admits a lifting over W_2 . Then the Frölicher spectral sequence of X degenerates at E_1 .*

The assumptions are fulfilled for curves, see Section 9. Moreover, if a smooth projective variety X in characteristic zero admits a model over $W(k)$ for some perfect field of characteristic $p \geq \dim X$ it follows from semi-continuity that the Frölicher spectral sequence of X degenerates at E_1 in characteristic zero. From this one obtains purely algebraic proofs of the following

Theorem 2.3. *Degeneration at E_1 holds for*

- (1) *smooth curves over arbitrary fields, and*

(2) *smooth projective varieties over fields of characteristic zero.*

We already mentioned above that varieties with global 1-forms that are not d -closed, e.g. the surfaces in [Mu61], provide examples where degeneration at E_1 does not hold.

Crystalline cohomology. To link deRham-, Betti- and Hodge-cohomology, we use *crystalline cohomology*. Its construction, due to Berthelot and Grothendieck, is quite involved [B-O]. This cohomology theory takes values in the Witt ring $W = W(k)$, which is a discrete valuation ring if k is perfect, see Section 9. If X is a smooth projective variety over a perfect field k then

- (1) the crystalline cohomology groups $H_{\text{cris}}^i(X/W)$ are finitely generated W -modules
- (2) they are zero for $i < 0$ and $i > 2 \dim(X)$
- (3) there are actions of Frobenius and Verschiebung on $H_{\text{cris}}^i(X/W)$
- (4) $H_{\text{cris}}^*(X/W) \otimes_W K$ satisfies Poincaré duality, where K denotes the field of fractions of W .
- (5) if X lifts over $W(k)$ then crystalline cohomology is isomorphic to the deRham cohomology of the lift, see Section 9.

We remind the reader that in order to get the “right” Betti numbers from the ℓ -adic cohomology groups $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$ we had to assume $\ell \neq p$. Crystalline cohomology takes values in $W(k)$ (recall $W(\mathbb{F}_p) \cong \mathbb{Z}_p$ with field of fractions \mathbb{Q}_p), and this is the “right” cohomology theory for $\ell = p$. In fact,

$$b_i(X) = \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^i(X, \mathbb{Q}_\ell) = \text{rank}_W H_{\text{cris}}^i(X/W), \quad \ell \neq p$$

i.e., the Betti numbers of X are encoded in the rank of crystalline cohomology. But since $H_{\text{cris}}^*(X/W)$ are W -modules, there may be non-trivial torsion - and this is precisely the explanation for the differences between Hodge- and Betti-numbers. More precisely, there is a universal coefficient formula, i.e., for all $m \geq 0$ there are short exact sequences

$$0 \rightarrow H_{\text{cris}}^m(X/W) \otimes_W k \rightarrow H_{\text{dR}}^m(X/k) \rightarrow \text{Tor}_1^W(H_{\text{cris}}^{m+1}(X/W), k) \rightarrow 0.$$

In particular, Betti- and deRham-numbers coincide if and only if the crystalline cohomology groups are torsion-free.

Finally, we note that Illusie [Ill79] constructed a complex $W\Omega_X^*$, the *deRham-Witt complex* and cohomology groups $H^j(X, W\Omega_X^i)$, all of which are W -modules. It is important to know that the torsion modules of these *Hodge-Witt cohomology groups* may not be finitely generated W -modules. However, there exists a spectral sequence

$$E_1^{i,j} = H^j(X, W\Omega_X^i) \Rightarrow H_{\text{cris}}^{i+j}(X/W)$$

the *slope spectral sequence*, which degenerates at E_1 modulo torsion. Computations and results for algebraic surfaces can be found in [Ill79, Section II.7].

3. Birational geometry of surfaces

From this section on, we study surfaces. To start with, we discuss the birational geometry of smooth surfaces, which turns out to be basically the same as over the complex numbers. Unless otherwise stated, results can be found in [Ba].

Riemann–Roch. Let S be a smooth projective surface over an algebraically closed field k of characteristic $p \geq 0$. Actually, asking for properness would be enough since by a theorem of Zariski and Goodman a surface that is smooth and proper over a field is automatically projective, see [Ba, Theorem 1.28].

Then we have *Noether's formula*

$$\chi(\mathcal{O}_S) = \frac{1}{12} (c_1^2(S) + c_2(S)) .$$

Moreover, if \mathcal{L} is a line bundle on S then we have the *Riemann–Roch formula*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_S) + \frac{1}{2} \mathcal{L} \cdot (\mathcal{L} \otimes \omega_S^\vee) .$$

Note that Serre duality holds over arbitrary fields, and thus $h^i(S, \mathcal{L}) = h^{2-i}(S, \mathcal{L})$. However, we have seen in Section 2 that Kodaira vanishing may not hold. Finally, if D is an effective divisor on S then D is a Gorenstein curve and the *adjunction formula* yields

$$\omega_D \cong (\omega_S \otimes \mathcal{O}_S(D))|_D ,$$

where ω_D and ω_S denote the respective dualizing sheaves. In particular, if D is irreducible we obtain

$$2p_a(D) - 2 = D^2 + K_S \cdot D .$$

We refer to [Hart, Chapter V.1], [Hart, Appendix A], [Ba, Chapter 5] and [Mil] for details and further references.

Blowing up and down. First of all, blowing up a point on a smooth surface has the same effect as in characteristic zero.

Proposition 3.1. *Let $f : \tilde{S} \rightarrow S$ be the blow-up in a closed point and denote by E the exceptional divisor on \tilde{S} . Then*

$$E \cong \mathbb{P}_k^1 \quad E^2 = -1, \quad \text{and} \quad K_{\tilde{S}} \cdot E = -1 .$$

Moreover, the equalities

$$b_2(\tilde{S}) = b_2(S) + 1 \quad \text{and} \quad \rho(\tilde{S}) = \rho(S) + 1$$

hold true.

As in the complex case we call such a curve E with $E^2 = -1$ and $E \cong \mathbb{P}^1$ an *exceptional (-1) -curve*. A surface that does not contain exceptional (-1) -curves is called *minimal*.

Conversely, exceptional (-1) -curves can be contracted and the proof (modifying a suitable hyperplane section) is basically the same as in characteristic zero, cf. [Ba, Theorem 3.30]

Theorem 3.2 (Castelnuovo). *Let E be an exceptional (-1) -curve on a smooth surface S . Then there exists a smooth surface S' and a morphism $f : S \rightarrow S'$, such that f is the blow-up of S' in a closed point with exceptional divisor E .*

Since b_2 drops every time one contracts an exceptional (-1) -curve, Castelnuovo's theorem immediately implies that for every surface S there exists a sequence of blow-downs $S \rightarrow S'$ onto a minimal surface S' . In this case, S' is called a *minimal model* of S .

Resolution of indeterminacy. As in characteristic zero, a rational map from a surface extends to a morphism after a finite number of blow-ups in closed points, which gives *resolution of indeterminacy* of a rational map. Moreover, every birational (rational) map can be factored as a sequence of blow-ups and Castelnuovo blow-downs, see, e.g., [Hart, Chapter V.5]. The proofs are the same as in characteristic zero.

Kodaira dimension. As over the complex numbers the following two notions are crucial in the Kodaira–Enriques classification of surfaces:

Definition 3.3. The n .th *plurigenus* $P_n(S)$ of a smooth projective surface S is defined to be

$$P_n(S) := h^0(S, \omega_S^{\otimes n}).$$

The *Kodaira dimension* $\kappa(S)$ is defined to be

$$\kappa(S) := \begin{cases} -\infty & \text{if } P_n(S) = 0 \text{ for all } n \geq 1 \\ 0 & \text{if } P_n(S) \text{ is bounded and } P_n(S) \neq 0 \text{ for some } n \geq 1 \\ 1 & \text{if } P_n(S) \text{ grows linearly as } n \rightarrow \infty \\ 2 & \text{if } P_n(S) \text{ grows quadratically as } n \rightarrow \infty \end{cases}$$

Note that the Riemann–Roch theorem implies that every surface possesses a well-defined Kodaira dimension. This introduced, we have the following deep result, which is already non-trivial in characteristic zero:

Theorem 3.4. *Let S be a smooth projective surface with $\kappa(S) \geq 0$. Then S possesses a unique minimal model.*

Birationally ruled surfaces. If a surface S is birational to $\mathbb{P}^1 \times C$ for some curve C , i.e., *birationally ruled*, then it satisfies $P_n(S) = 0$ for all $n \geq 1$ and is thus of Kodaira dimension $\kappa(S) = -\infty$. Conversely, one can show that there exists a smooth rational curve on S that moves if $\kappa(S) = -\infty$ and thus

Theorem 3.5. *If $\kappa(S) = -\infty$ then S is birationally ruled, i.e., birational to $\mathbb{P}^1 \times C$, where C is a smooth curve. In this case, we have*

$$h^1(S, \mathcal{O}_S) = \frac{1}{2}b_1(S) = g(C),$$

where $g(C)$ denotes the genus of C .

As in the complex case, minimal models for surfaces with $\kappa(S) = -\infty$ are not unique. More precisely, we have Nagata's result

Theorem 3.6. *Let S be a minimal surface with $\kappa(S) = -\infty$.*

- (1) *If $h^1(S, \mathcal{O}_S) \geq 1$ then the image C of the Albanese map is a smooth curve. Moreover, there is a rank two vector bundle \mathcal{E} on C such that $\text{alb}_S : S \rightarrow C$ is isomorphic to $\mathbb{P}(\mathcal{E}) \rightarrow C$.*
- (2) *If $h^1(S, \mathcal{O}_S) = 0$ then S is isomorphic to \mathbb{P}^2 or a Hirzebruch surface $\mathbb{F}_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \rightarrow \mathbb{P}^1$ with $d \neq 1$.*

Also, Castelnuovo’s cohomological characterization of rational surfaces holds true and the proof for positive characteristics is due to Zariski [Za58]:

Theorem 3.7 (Castelnuovo–Zariski). *For a smooth projective surface S the following are equivalent*

- (1) *S is rational, i.e., birational to \mathbb{P}^2 ,*
- (2) *$h^1(S, \mathcal{O}_S) = P_2(S) = 0$*
- (3) *$b_1(S) = P_2(S) = 0$.*

So far, things look pretty much the same as over the complex numbers. However, one has to be a little bit careful with the notion of uniruledness: we will see in Section 8 below that unirationality (resp. uniruledness) does *not* imply rationality (resp. ruledness).

4. (Quasi)-elliptic fibrations

For the classification of surfaces of special type in characteristic zero, elliptic fibrations play an important role. In positive characteristic, *wild fibers* and *quasi-elliptic fibrations* are new features that show up. We refer to [Ba, Section 7] for an introduction, [B-M2], [B-M3] for more details, as well as [C-D, Chapter V] for more specialized topics.

Quasi-elliptic fibrations. Given a dominant morphism from a smooth surface S onto a curve in any characteristic, we may pass to its Stein factorization and obtain a fibration $S \rightarrow B$, cf. [Hart, Corollary III.11.5]. Then its generic fiber S_η is an integral curve, i.e., reduced and irreducible. Moreover, in characteristic zero Bertini’s theorems imply that S_η is in fact smooth over the function field $k(B)$. Now, if $\text{char}(k) = p > 0$, then it is still true that the generic fiber is a regular curve, i.e., all local rings are regular local rings. However, this does not imply that S_η is smooth over $k(B)$. Note that since S_η is one-dimensional regularity is the same as normality. See [Mat80, Chapter 11.28] for a discussion of smoothness and regularity.

Suppose S_η is not smooth over $k(B)$ and denote by $\overline{k(B)}$ the algebraic closure of $k(B)$. Then $S_{\overline{\eta}} := S_\eta \otimes_{k(B)} \overline{k(B)}$ is still reduced and irreducible [Ba, Theorem 7.1] but no longer regular and we denote by $S'_{\overline{\eta}} \rightarrow S_{\overline{\eta}}$ the normalization morphism. Then Tate’s theorem on genus change in inseparable extensions [Ta52] (see [Sch09] for a modern treatment) states

Theorem 4.1 (Tate). *Under the previous assumptions, the normalization map $S'_{\overline{\eta}} \rightarrow S_{\overline{\eta}}$ is a homeomorphism, i.e., $S_{\overline{\eta}}$ has unibranch singularities (“cusps”). Moreover, if $p \geq 3$, then the arithmetic genus of every cusp of $S_{\overline{\eta}}$ is divisible by $(p - 1)/2$.*

If the generic fiber S_η has arithmetic genus one, and the fiber is not smooth, then the normalization of S_η is \mathbb{P}_k^1 and the singularity is one cusp of arithmetic genus one. Since $(p-1)/2$ divides this genus if $p \geq 3$, we find $p = 3$ as only solution. Thus:

Corollary 4.2. *Let $f : S \rightarrow B$ be a fibration from a smooth surface whose generic fiber S_η is a curve of arithmetic genus one, i.e., $h^1(S_\eta, \mathcal{O}_{S_\eta}) = 1$. Then*

- (1) *either S_η is smooth over $k(B)$,*
- (2) *or S_η is a singular rational curve with one cusp.*

The second case can happen in characteristic 2 and 3 only.

Definition 4.3. If the generic fiber of a fibration $S \rightarrow B$ is a smooth curve of genus one, the fibration is called *elliptic*. If the generic fiber is a curve that is not smooth over $k(B)$, the fibration is called *quasi-elliptic*, which can exist in characteristic 2 and 3 only.

We refer to [B-M2] and [Ba, Exercises 7.5 and 7.6] for examples of quasi-elliptic fibrations and to [B-M3] for results about the geometry of quasi-elliptic fibrations. For results on quasi-elliptic fibrations in characteristic 3 see [La79].

We note that quasi-elliptically fibered surfaces are always uniruled, but may not be birationally ruled and refer to Section 8 where we discuss this phenomenon in detail.

We note that the situation gets worse in higher dimensions: Mori and Saito [M-S03] constructed Fano 3-folds X in characteristic 2 together with fibrations $X \rightarrow S$, whose generic fibers are conics in $\mathbb{P}_{k(S)}^2$ that become non-reduced over $\overline{k(S)}$. Such fibrations are called *wild conic bundles*.

Canonical bundle formula. Let S be a smooth surface and $f : S \rightarrow B$ be an elliptic or quasi-elliptic fibration. Since B is smooth, we obtain a decomposition

$$R^1 f_* \mathcal{O}_S = \mathcal{L} \oplus \mathcal{T}$$

where \mathcal{L} is a line bundle and \mathcal{T} is a torsion sheaf on B . In characteristic zero, the torsion sheaf \mathcal{T} is trivial.

Definition 4.4. Let $b \in B$ a point of the support of \mathcal{T} . Then the fiber of f above b is called a *wild fiber* or an *exceptional fiber*.

Proposition 4.5. *Let $f : S \rightarrow B$ be a quasi-elliptic fibration, $b \in B$ and F_b the fiber above b . Then the following are equivalent*

- (1) *$b \in \text{Supp}(\mathcal{T})$, i.e., F_b is a wild fiber,*
- (2) *$h^1(F_b, \mathcal{O}_{F_b}) \geq 2$,*
- (3) *$h^0(F_b, \mathcal{O}_{F_b}) \geq 2$.*

In particular, wild fibers are multiple fibers. Moreover, if F_b is a wild fiber then its multiplicity is divisible by p and $h^1(S, \mathcal{O}_S) \geq 1$ holds true.

The canonical bundle formula for relatively minimal (quasi-)elliptic fibrations has been proved in [B-M2] - as usual, relatively minimal means that there are no exceptional (-1) -curves in the fibers of the fibration.

Theorem 4.6 (Canonical bundle formula). *Let $f : S \rightarrow B$ be a relatively minimal (quasi-)elliptic fibration from a smooth surface. Then*

$$\omega_S \cong f^*(\omega_B \otimes \mathcal{L}^\vee) \otimes \mathcal{O}_S \left(\sum_i a_i P_i \right),$$

where

- (1) $m_i P_i = F_i$ are the multiple fibers of f
- (2) $0 \leq a_i < m_i$
- (3) $a_i = m_i - 1$ if F_i is not a wild fiber
- (4) $\deg(\omega_S \otimes \mathcal{L}^\vee) = 2g(B) - 2 + \chi(\mathcal{O}_S) + \text{length}(T)$.

For more results on the a_i 's we refer to [C-D, Proposition V.5.1.5], as well as to [K-U85] for more on wild fibers.

Degenerate fibers of (quasi-)elliptic fibrations. Usually, an elliptic fibration has fibers that are not smooth and the possible cases have been classified by Kodaira and Néron. The list is the same as in characteristic zero, cf. [C-D, Chapter V, §1] and [Sil94, Theorem IV.8.2]. This should not be such a surprise, as the classification of degenerate fibers rests on the adjunction formula and on matrices of intersection numbers.

Let us recall that the possible fibers together with their Kodaira symbols are as follows (after reduction):

- (1) an irreducible rational curve with a node as singularity (I_1)
- (2) a cycle of $n \geq 2$ rational curves (I_n)
- (3) an irreducible rational curve with a cusp as singularity (II)
- (4) a configuration of rational curves forming a root system of type A_2^* (III), \bar{A}_3 (IV), \tilde{E}_6 (IV*), \tilde{E}_7 (III*), \tilde{E}_8 (II*) or \tilde{D}_n (I_{n-4}^*)

In the first two cases, the reduction is called *multiplicative* or *semi-stable*, whereas in the last two cases it is called *additive* or *unstable*. The names multiplicative and additive come from the theory of Néron models [Sil94, Chapter IV]. The other names are explained by the fact that semi-stable reduction remains semi-stable after pull-back, whereas unstable reduction may become semi-stable after pull-back. In fact, for every fiber with unstable reduction there exists a pull-back making the reduction semi-stable [Sil94, Proposition IV.10.3].

For an elliptic fibration $S \rightarrow B$ from a smooth surface, the second Chern class (Euler number), can be expressed in terms of the singular fibers by *Ogg's formula*:

$$c_2(S) = \sum_i \nu(\Delta_i)$$

where i runs through the singular fibers, Δ_i is the *minimal discriminant* of the singular fiber and ν denotes its valuation. If a fiber has n irreducible components then this minimal discriminant is as follows

$$\nu(\Delta) = \begin{cases} 1 + (n - 1) & \text{if the reduction is multiplicative, i.e., of type } I_n \\ 2 + (n - 1) + \delta & \text{if the reduction is additive} \end{cases}$$

Here, δ is the *Swan conductor* or *wild part of the conductor* of the fiber, which is zero if $p \neq 2, 3$. We refer to [Sil94, Chapter IV, §10] for details and to [C-D, Proposition 5.1.6] for a version for quasi-elliptic fibrations.

The generic geometric fiber of a quasi-elliptic fibration is a singular rational curve with a cusp. Fibers are reduced or have multiplicity equal to the characteristic $p = 2, 3$. The list of possible degenerate fibers is as follows [C-D, Corollary 5.2.4]:

$$\begin{aligned} p = 3 & \text{ II, IV, IV}^* \text{ and II}^* \\ p = 2 & \text{ II, III, III}^*, \text{II}^* \text{ and I}_n^*. \end{aligned}$$

Whenever a (quasi-)elliptic fibration has a section, there exists a *Weierstraß model* [C-D, Chapter 5, §5], which is slightly more involved in characteristic 2, 3 than in the other positive characteristics or characteristic zero.

5. Enriques–Kodaira classification

We now come to the Kodaira–Enriques classification of surfaces, which has been established by Bombieri and Mumford in [Mu69], [B-M2] and [B-M3]. So let S be a smooth projective surface of Kodaira dimension $\kappa(S)$. We have already seen that in Section 3 that

Theorem 5.1. *If $\kappa(S) = -\infty$ then S is birationally ruled.*

In fact, $\kappa(S) = -\infty$ is equivalent to $p_{12}(S) = 0$, where p_{12} is the 12.th plurigenus [Ba, Theorem 9.8]. Moreover, although minimal models are not unique they have the same structure as in characteristic zero, see Section 3. In Section 8 we shall see that uniruled surfaces in positive characteristic may *not* fulfill $\kappa = -\infty$.

Definition 5.2. The *canonical ring* $R_{\text{can}}(S)$ of a surface S is defined to be

$$R_{\text{can}}(S) := \bigoplus_{n \geq 0} H^0(S, \omega_S^{\otimes n})$$

Theorem 5.3 (Zariski–Mumford). *The canonical ring $R_{\text{can}}(S)$ of a smooth projective surface is a finitely generated k -algebra. If $\kappa(S) \geq 0$ then $R_{\text{can}}(S)$ has transcendence degree $1 + \kappa(S)$ over k .*

We refer to [B-M2] and [Ba, Corollary 9.10]. More generally, we refer to [Ba, Chapter 14] for a discussion of Zariski decompositions and finite generation of the more general rings $R(S, D)$, where D is \mathbb{Q} -divisor on S .

Kodaira dimension one. For a surface with $\kappa(S) \geq 1$ one studies the *Iitaka-fibration*

$$S \dashrightarrow \text{Proj } R_{\text{can}}(S).$$

By the theorem of Zariski–Mumford just mentioned, the right hand side is a projective variety of dimension $\kappa(S)$.

Theorem 5.4. *Let S be a minimal surface with $\kappa(S) = 1$. Then (the Stein factorization of) the Iitaka fibration is a morphism, which is a relatively minimal elliptic or quasi-elliptic fibration.*

If $\kappa(S) = 1$ and $p \neq 2, 3$ then the elliptic fibration is unique, $|mK_S|$ for $m \geq 14$ defines this elliptic fibration, and 14 is the optimal bound [K-U85]. The main difficulties solved by this paper come from the existence of wild fibers. We note that this bound $m \geq 14$ is better than Iitaka's bound $m \geq 86$ over the complex numbers, since over the complex numbers also analytic surfaces that are not algebraic have to be taken into account.

Kodaira dimension two. Although we will come back to surfaces of general type in Section 7 let us already mention the following result.

Theorem 5.5. *Let S be a minimal surface with $\kappa(S) = 2$, i.e., a surface of general type. Then the Iitaka fibration is a birational morphism that contracts all (-2) -curves on S .*

6. Kodaira dimension zero

As in the complex case, surfaces of Kodaira dimension zero fall into four classes. However, there is a new class of Enriques surfaces in characteristic 2, as well as the new class of quasi-hyperelliptic surfaces in characteristic 2 and 3. In particular, there are no new classes in characteristic $p \geq 5$.

We start with a result that follows from the explicit classification, especially of the (quasi-)hyperelliptic surfaces (discussed below):

Theorem 6.1. *Let S be a minimal surface with $\kappa(S) = 0$. Then $\omega_S^{\otimes 12} \cong \mathcal{O}_S$.*

Let us now discuss the four classes in greater detail:

Abelian surfaces. These are two-dimensional Abelian varieties. Their main invariants are as in characteristic zero:

$$\begin{array}{llll} \omega_S \cong \mathcal{O}_S & p_g = 1 & h^{0,1} = 2 & h^{1,0} = 2 \\ \chi(\mathcal{O}_S) = 1 & c_2 = 0 & b_1 = 4 & b_2 = 2 \end{array}$$

Abelian surfaces are usually studied within the framework of Abelian varieties of arbitrary dimension. We note that there exists a huge amount of literature on Abelian varieties and their moduli spaces, both in characteristic zero and in positive characteristic, see, e.g., [Mu70].

By an (unpublished) result of Grothendieck, Abelian varieties lift to characteristic zero, see also Section 9.

For an Abelian variety A of dimension g multiplication by p is a finite morphism and its kernel $A[p]$ is a finite and flat group scheme of length p^{2g} . Its identity component $A[p]^\circ$ is infinitesimal of length at least g . The quotient $A[p]/A[p]^\circ$ is an étale group scheme isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$, for some $0 \leq r \leq g$. This quantity r is called the p -rank of A . For Abelian varieties of dimension at most two, the p -rank can be detected by the action $F : H^1(A, \mathcal{O}_A) \rightarrow H^1(A, \mathcal{O}_A)$ of Frobenius. (This is not true in higher dimensions: if F acts as zero then all Newton slopes are positive, but the Newton polygon may not be a straight line.)

Definition 6.2. An Abelian surface A is called

- (1) *ordinary* if $r = 2$. Equivalently, F acts bijectively on $H^1(A, \mathcal{O}_A)$.

(2) *supersingular* if $r = 0$. Equivalently, F is zero on $H^1(A, \mathcal{O}_A)$.

We note that the image of the Albanese morphism of a uniruled surface is at most one-dimensional. Thus, an Abelian surface cannot be uniruled, which we note here in view of Shioda's notion of supersingularity and its connection to unirationality discussed in Section 8.

K3 surfaces. These surfaces are characterized by the following invariants:

$$\begin{array}{llll} \omega_S \cong \mathcal{O}_S & p_g = 1 & h^{0,1} = 0 & h^{1,0} = 0 \\ \chi(\mathcal{O}_S) = 2 & c_2 = 24 & b_1 = 0 & b_2 = 22 \end{array}$$

The next result is the Bogomolov–Tian–Todorov unobstructedness theorem for K3 surfaces in positive characteristic:

Theorem 6.3 (Rudakov–Šafarevič). *A K3 surface has no global vector fields. Thus,*

$$H^2(S, \Omega_S) \xrightarrow{SD} H^0(S, \Theta_S \otimes \omega_S) = H^0(S, \Theta_S) = 0$$

where SD denotes Serre duality. In particular, the deformations of these surfaces are unobstructed.

Over the complex numbers this follows from the Hodge symmetry $h^{1,2} = h^{0,2}$, which is induced by complex conjugation and thus may not hold over arbitrary ground fields. The proof in positive characteristic of [R-S76] uses characteristic- p -techniques: from the existence of a hypothetical global vector field on a K3 surface S , they deduce the existence of a purely inseparable morphism from S to some other surface S' , see Section 11. A careful analysis of this hypothetical inseparable morphism and of S' finally yields the desired contradiction.

We note that over fields of positive characteristic and in dimension three the Bogomolov–Tian–Todorov unobstructedness theorem for Calabi–Yau varieties fails, cf. [Hir99a] and [Sch04].

Coming back to K3 surfaces this result implies that they possess formal liftings and Deligne [Del81] showed in fact (see also Appendix 9):

Theorem 6.4 (Deligne). *K3 surfaces lift to characteristic zero.*

We come back to K3 surfaces in Section 8, where we discuss conjectures that try to characterize the unirational ones.

Enriques surfaces. In characteristic $p \neq 2$ these surfaces are characterized by the following invariants:

$$\begin{array}{llll} \omega_S \not\cong \mathcal{O}_S & \omega_S^{\otimes 2} \cong \mathcal{O}_S & p_g = 0 & h^{0,1} = 0 \\ \chi(\mathcal{O}_S) = 1 & c_2 = 12 & b_1 = 0 & b_2 = 10 \end{array}$$

Moreover, the canonical sheaf ω_S defines a double cover $\tilde{S} \rightarrow S$, where S is a K3 surface. Also, Enriques surfaces always possess elliptic or quasi-elliptic fibrations.

The most challenging case is characteristic 2, where Enriques surfaces are characterized by (\equiv denotes numerical equivalence)

$$\omega_S \equiv \mathcal{O}_S \quad \chi(\mathcal{O}_S) = 1 \quad c_2 = 12 \quad b_1 = 0 \quad b_2 = 10$$

It turns out that $p_g = h^{0,1} \leq 1$, see [B-M3]. Since $b_1 = 0$ we conclude that the Picard scheme of an Enriques surface with $h^{0,1} = 1$ is not smooth. In this case, Frobenius induces a map $F : H^1(S, \mathcal{O}_S) \rightarrow H^1(S, \mathcal{O}_S)$, which is either identically zero or bijective. We thus obtain three possibilities:

Definition 6.5. An Enriques surface in characteristic 2 is called

- (1) *classical* if $h^{0,1} = p_g = 0$, and
- (2) *non-classical* if $h^{0,1} = p_g = 1$.

Moreover, a non-classical Enriques surface is called

- (1) *ordinary* if Frobenius acts bijectively on $H^1(S, \mathcal{O}_S)$, and
- (2) *supersingular* if Frobenius is zero on $H^1(S, \mathcal{O}_S)$.

All three types exist [B-M3]. Note that the naming of the two types of non-classical surfaces is inspired from Abelian surfaces.

Moreover, all Enriques surfaces possess elliptic or quasi-elliptic fibrations. These fibrations always have multiple fibers. Moreover, if S is classical every (quasi-)elliptic fibration has precisely two multiple fibers, both of multiplicity two and neither of them is wild. If S non-classical then there is only one multiple fiber, which is wild of multiplicity two. Finally, if S is non-classical and ordinary it does not possess quasi-elliptic fibrations. We refer to [C-D, Chapter V.7] for details.

As explained in [B-M3, §3], all Enriques surfaces possess a finite and flat morphism of degree two

$$\varphi : \tilde{S} \rightarrow S$$

such that $\omega_{\tilde{S}} \cong \mathcal{O}_S$, $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) = h^2(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 1$, and $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$, i.e., \tilde{S} is “K3-like”.

If S is non-classical and ordinary then φ is an étale morphism of degree two and \tilde{S} is in fact a K3 surface. In the other cases, however, \tilde{S} is usually only an integral Gorenstein surface that need not even be normal since φ is a torsor under a infinitesimal group scheme. This makes the classification of Enriques surfaces in characteristic 2 so difficult.

We refer to [B-M3], [La83b], [La88] and, of course, to [C-D] for more details and partial classification results.

(Quasi-)hyperelliptic surfaces. In characteristic $p \neq 2, 3$ these surfaces are characterized by the following invariants:

$$\begin{array}{llll} \omega_S \not\cong \mathcal{O}_S & \omega_S^{\otimes 12} \cong \mathcal{O}_S & p_g = 0 & h^{0,1} = 0 \\ \chi(\mathcal{O}_S) = 0 & c_2 = 0 & b_1 = 2 & b_2 = 2 \end{array}$$

Moreover, these surfaces are equipped with two elliptic fibrations: one is the Albanese fibration $S \rightarrow E$, where E is an elliptic curve, and then there exists a second fibration $S \rightarrow \mathbb{P}^1$. It turns out that all these surfaces arise as quotients

$$S = (E \times F)/G$$

where E and F are elliptic curves, G is a group acting faithfully on E and F and the quotient yielding S is via the diagonal action. In particular, the classical list [Ba, List 10.27] of Bagnera–DeFranchis gives all classes.

The more complicated classes arise in characteristic 2 and 3. Firstly, these surfaces are characterized by (again, \equiv denotes numerical equivalence)

$$\omega_S \equiv \mathcal{O}_S \quad \chi(\mathcal{O}_S) = 0 \quad c_2 = 0 \quad b_1 = 2 \quad b_2 = 2$$

It turns out that $1 \leq p_g + 1 = h^{0,1} \leq 2$, and the surfaces with $h^{0,1} = 2$ are those with non-smooth Picard scheme **[B-M3]**.

In any case, the Albanese morphism $S \rightarrow \text{Alb}(S)$ is onto an elliptic curve with generic fiber a curve of genus one, which motivates the following

Definition 6.6. The surface is called *hyperelliptic*, if $S \rightarrow \text{Alb}(S)$ is an elliptic fibration, and *quasi-hyperelliptic* if this fibration is quasi-elliptic.

In both cases, there exists a second fibration $S \rightarrow \mathbb{P}^1$, which is always elliptic. Finally, for every (quasi-)hyperelliptic surface S , there exists

- (1) an elliptic curve E ,
- (2) a curve C of arithmetic genus one, which is smooth if S is hyperelliptic, or rational with a cusp if S is quasi-hyperelliptic
- (3) a finite and flat group scheme G (possibly non-reduced), together with embeddings $G \rightarrow \text{Aut}(C)$ and $G \rightarrow \text{Aut}(E)$, where G acts by translations on E

Then S can be described as

$$S = (E \times C)/G,$$

the Albanese map arises as projection onto E/G with fiber C and the other fibration onto $C/G \cong \mathbb{P}^1$ is elliptic with fiber E .

It turns out that G may contain infinitesimal subgroups. Even in the hyperelliptic case we obtain new cases since the automorphism groups of elliptic curves with $j = 0$ in characteristic 2 and 3 is larger than in any other characteristic, see Section 1. We refer to **[B-M2]** for the complete classification of hyperelliptic surfaces and to **[B-M3]** for the classification of quasi-hyperelliptic surfaces.

An interesting feature in characteristic 2 and 3 is the possibility that G acts trivially on $\omega_{E \times C}$, and thus the canonical sheaf on S is trivial. In this case we get $p_g = 1$, $h^{0,1} = 2$ and the Picard scheme of S is not reduced.

7. General type

Pluricanonical maps. Let S be a minimal surface of general type. Clearly,

$$K_S^2 > 0$$

since some pluricanonical map has a two-dimensional image. We shall see below that Castelnuovo's inequality $c_2 > 0$ may fail. Let us recall that a *rational* (-2) -curve is a curve C on a surface with $C \cong \mathbb{P}_k^1$ and $C^2 = -2$. We discuss in Section 10 in greater detail that configurations of rational (-2) -curves with negative definite intersection matrix can be contracted to DuVal singularities.

Theorem 7.1. *Let S be a minimal surface of general type. Then the (a priori) rational map to the canonical model*

$$S \dashrightarrow S_{\text{can}} := \text{Proj } R_{\text{can}} = \text{Proj } \bigoplus_{n \geq 0} H^0(S, \omega_S^{\otimes n})$$

is a birational morphism that contracts all (-2) -curves and nothing more.

Bombieri's results on pluri-canonical systems have been extended to positive characteristic in [Ek88] and refined in [SB91a], and we refer to these articles for more results. We give a hint of how to modify the classical proofs below. Also, the reader who is puzzled by the possibility of purely inseparably uniruled surfaces of general type in the statements below is advised to have a short look at Section 8 first.

Theorem 7.2 (Ekedahl, Shepherd-Barron). *Let S be a minimal surface of general type and consider the linear system $|mK_S|$ on the canonical model S_{can}*

- (1) *it is ample for $m \geq 5$ or if $m = 4$ and $K_S^2 \geq 2$ or $m = 3$ and $K_S^2 \geq 3$,*
- (2) *it is base-point free for $m \geq 4$ or if $m = 3$ and $K_S^2 \geq 2$,*
- (3) *it is base-point free for $m = 2$ if $K_S^2 \geq 5$ and $p \geq 11$ or $p \geq 3$ and S is not inseparably uniruled,*
- (4) *it defines a birational morphism for $m = 2$ if $K_S^2 \geq 10$, S has no pencil of genus 2 curves and $p \geq 11$ or $p \geq 5$ and S is not purely inseparably uniruled.*

There is a version of Ramanujam-vanishing, see [Ek88, Theorem II.1.6] for the precise statement:

Theorem 7.3 (Ekedahl). *Let S be a minimal surface of general type and let \mathcal{L} be a line bundle that is numerically equivalent to $\omega_S^{\otimes i}$ for some $i \geq 1$. Then $H^1(S, \mathcal{L}^\vee) = 0$ except possibly for certain surfaces in characteristic 2 with $\chi(\mathcal{O}_S) \leq 1$.*

On the other hand, minimal surfaces of general type with $H^1(S, \omega_S^\vee) \neq 0$ in characteristic 2 are constructed in [Ek88, Proposition I.2.14].

Bombieri's proof of the above results over the complex numbers is based on vanishing theorems $H^1(S, \mathcal{L}) = 0$ for certain more or less negative line bundles. However, these vanishing results do not hold in positive characteristic, see [Ra78] for explicit counter-examples. Ekedahl's strategy developed in [Ek88] overcomes this difficulty as follows: He considers a line bundle \mathcal{L} and its Frobenius-pullback $F^*(\mathcal{L}) \cong \mathcal{L}^{\otimes p}$ as group schemes over S . Then Frobenius induces a short exact sequence of group schemes (for the flat topology on S)

$$(1) \quad 0 \rightarrow \alpha_{\mathcal{L}} \rightarrow \mathcal{L} \xrightarrow{F} F^*(\mathcal{L}) \rightarrow 0,$$

where $\alpha_{\mathcal{L}}$ is defined to be the kernel of F , see also Section 1 for the definition of the finite and flat group scheme α_p . This $\alpha_{\mathcal{L}}$ is an infinitesimal group scheme over S and can be thought of as a possibly non-trivial family of α_p 's over S .

Now, if \mathcal{L} is ample then $H^1(S, \mathcal{L}^{\otimes \nu}) = 0$ for $\nu \gg 0$, see [Hart, Theorem III.7.6]. In order to get vanishing of $H^1(S, \mathcal{L})$ we assume that this is not the case

and replace \mathcal{L} by some $\mathcal{L}^{\otimes \nu}$ such that $H^1(S, \mathcal{L}) \neq 0$ and $H^1(S, \mathcal{L}^{\otimes p}) = 0$. Then the long exact sequence in cohomology for (1) yields

$$H_{\text{fl}}^1(S, \alpha_{\mathcal{L}}) \neq 0.$$

Such a cohomology class corresponds to an $\alpha_{\mathcal{L}}$ -torsor, which implies that there exists a purely inseparable morphism of degree p

$$Y \xrightarrow{\pi} S$$

where Y is an integral Gorenstein surface, whose dualizing sheaf satisfies $\omega_Y \cong \pi^*(\omega_S \otimes \mathcal{L}^{p-1})$. For example, suppose S is of general type and $\mathcal{L} = \omega_S^{\otimes (-m)}$ for some $m \geq 2$. Then *either* $H^1(S, \mathcal{L}) = 0$ and one proceeds as in the classical case *or* there exists an inseparable cover $Y \rightarrow S$, where ω_Y is negative. The second alternative implies that S is inseparably uniruled, and a further analysis of the situation leads to a contradiction or an explicit counter-example for a vanishing result.

Castelnuovo's inequality. In [Ra78], minimal surfaces of general type with $c_2 < 0$ for all characteristics $p \geq 5$ are constructed. However, there is the following structure result:

Theorem 7.4 (Shepherd-Barron [SB91b]). *Let S be a minimal surface of general type.*

- (1) *If $c_2(S) = 0$ then S is inseparably dominated by a surface of special type.*
- (2) *If $c_2(S) < 0$ then the Albanese map $S \rightarrow \text{Alb}(S)$ has one-dimensional image with generic fiber a singular rational curve.*

In characteristic $p \geq 11$, surfaces of general type fulfill $\chi(\mathcal{O}_S) > 0$.

We refer to [SB91b] for more precise statements and note that is still unknown whether there exist surfaces of general type with $\chi(\mathcal{O}_S) \leq 0$ in characteristic $p \leq 7$.

Noether's inequality. Minimal surfaces of general type fulfill *Noether's inequality*

$$K_S^2 \geq 2p_g(S) - 4$$

Moreover, the classification of *Horikawa surfaces*, i.e., minimal surfaces of general type that fulfill $K^2 = 2p_g - 4$, is the same as over the complex numbers: all of them are double covers of rational surfaces via their 1-canonical map [Li08]. In characteristic 2 it can happen that the 1-canonical map becomes purely inseparable, and the corresponding Horikawa surfaces are unirational, see also Section 8. Unirational Horikawa surfaces in characteristic $p \geq 3$ have been constructed in [L-S09].

Bogomolov–Miyaoka–Yau inequality. A minimal surface of general type over the complex numbers fulfills $K_S^2 \leq 9\chi(\mathcal{O}_S)$ or, equivalently, $K_S^2 \leq 3c_2(S)$. This is proved using analytic methods from differential geometry. Moreover, by a result of Yau, surfaces with $c_1^2 = 3c_2$ are uniformized by the complex 2-ball and thus these surfaces are rigid by a result Siu.

This inequality is known to fail in positive characteristic, see, for example, [Sz79, Section 3.4.1]. There are beautiful counter-examples in [BHH87, Kapitel 3.5.J], where covers of \mathbb{P}^2 ramified over special line configurations that only exist in positive characteristic are used. Similar constructions appeared in [Ea08].

In [Ek88, Remark (i) to Proposition 2.14], a 10-dimensional family of surfaces with $K^2 = 9$ and $\chi(\mathcal{O}_S) = 1$ in characteristic 2 is constructed, i.e., rigidity on the Bogomolov–Miyaoka–Yau line fails.

On the other hand there is the following positive result [SB91a]

Theorem 7.5 (Shepherd-Barron). *If S is a minimal surface of general type in characteristic 2 that lifts over $W_2(k)$ then $c_1^2(S) \leq 4c_2(S)$ holds true.*

We refer to [SB91a] for results under what assumptions Bogomolov’s inequality $c_1^2(\mathcal{E}) \leq 4c_2(\mathcal{E})$ holds for stable rank 2 vector bundles.

Global vector fields. The tangent space to the automorphism group scheme of a smooth variety is isomorphic to the space of global vector fields. Since a surface of general type has only finitely many automorphisms this implies that there are no global vector fields on a surface of general type in characteristic zero. However, in positive characteristic, the automorphism group scheme of a surface of general type is still finite but may contain infinitesimal subgroup schemes, which have non-trivial tangent spaces. Thus, infinitesimal automorphism group schemes of surfaces of general type in positive characteristic give rise to non-trivial global vector fields. For examples, we refer to [La83a].

Non-classical Godeaux surfaces. Since K^2 of a minimal surface of general type is always positive, one can ask for surfaces with $K_S^2 = 1$. It turns out that these fulfill $1 \leq \chi(\mathcal{O}_S) \leq 3$ and thus the lowest invariants possible are as follows:

Definition 7.6. A *numerical Godeaux surface* is a minimal surface of general type with $\chi(\mathcal{O}_S) = K_S^2 = 1$. Such a surface is called *classical* if $p_g = h^{0,1} = 0$ and otherwise *non-classical*.

In characteristic zero or in characteristic $p \geq 7$ it turns out that numerical Godeaux surfaces are classical [Li09b]. Also, quotients of a quintic surface in \mathbb{P}^3 by a free $\mathbb{Z}/5\mathbb{Z}$ -action provide examples of classical Godeaux surfaces in characteristic $p \neq 5$. Classical and non-classical Godeaux surfaces in characteristic $p = 5$ have been constructed in [La81] and [Mir84]. Non-classical Godeaux surfaces in characteristic $p = 5$ have been completely classified [Li09b] - it turns out that all of them arise as quotients of (possibly highly singular) quintic surfaces in \mathbb{P}^3 by $\mathbb{Z}/5\mathbb{Z}$ or α_p . We finally note that non-classical Godeaux surfaces are those with non-reduced Picard schemes.

Quite generally, in [Li09a] it has been shown that given n there exists an integer $P(n)$ such that minimal surfaces of general type with $K^2 \leq n$ in characteristic $p \geq P(n)$ have a reduced Picard scheme. Thus $P(1) = 7$ but $P(n)$ is unbounded as n gets larger and larger.

8. Unirationality and supersingularity

From this section on we discuss more specialized topics. We start with questions related to unirationality. More precisely, we circle around rationality, unirationality, their effect on Néron–Severi groups, and the formal Brauer group. We discuss these for K3 surfaces and for surfaces that are defined over finite fields.

An instructive observation. To start with, let $\varphi : X \dashrightarrow Y$ be a dominant and generically finite morphism in characteristic zero. Then the pull-back of a non-trivial pluricanonical form is again a non-trivial pluricanonical form. Thus, if $\kappa(X) = -\infty$, also $\kappa(Y) = -\infty$ holds true. However, over a field of positive characteristic p , the example

$$\varphi : t \mapsto t^p, \quad \text{and then} \quad \varphi^*(dt) = dt^p = pt^{p-1}dt = 0$$

shows that the pull-back of a non-trivial pluricanonical form may very well become trivial after pullback. In particular, the previous characteristic zero argument showing that the Kodaira dimension cannot increase under generically finite morphisms breaks down.

Zariski surfaces. In fact, Zariski [Za58] has given the first examples of unirational surfaces in positive characteristic that are not rational: for a generic choice of a polynomial $f(x, y)$ of sufficiently large degree,

$$(2) \quad z^p - f(x, y) = 0$$

extends to an inseparable cover $X \rightarrow \mathbb{P}^2$, where X has “mild” singularities and where usually $\kappa(\tilde{X}) \geq 0$ for some resolution of singularities $\tilde{X} \rightarrow X$. Moreover, we have an inclusion of function fields

$$k(x, y) \subset k(\tilde{X}) = k(x, y)[\sqrt[p]{f(x, y)}] \subset k(\sqrt[p]{x}, \sqrt[p]{y})$$

i.e., \tilde{X} is unirational. Surfaces of the form (2) are called *Zariski-surfaces*.

Theorem 8.1. (*Zariski*) *In every positive characteristic there exist unirational surfaces that are not rational.*

However, we remind the reader from Section 3 that rational surfaces are still characterized as those surfaces satisfying $h^{0,1} = p_2 = 0$.

Quasi-elliptic surfaces. If $S \rightarrow B$ is a quasi-elliptic fibration from a surface S with generic fiber S_η , then there exists a purely inseparable extension $L/k(B)$ of degree $p = \text{char}(k)$ such that $S_\eta \otimes_{k(B)} L$ is not normal, i.e., the cusp “appears” over L , see [B-M3]. Thus, the normalization of $S \otimes_{k(B)} L$ is a birationally ruled surface over B and we get the following result

Theorem 8.2. *Let S be a surface and $S \rightarrow B$ be a quasi-elliptic fibration. Then there exists a purely inseparable and dominant rational map $B \times \mathbb{P}^1 \dashrightarrow S$, i.e., S is (purely inseparably) uniruled.*

In particular, if $S \rightarrow \mathbb{P}^1$ is a quasi-elliptic fibration then S is a Zariski surface, and in particular unirational.

Fermat surfaces. If the characteristic $p = \text{char}(k)$ does not divide n then the Fermat surface S_n , i.e., the hypersurface

$$S_n := \{x_0^n + x_1^n + x_2^n + x_3^n = 0\} \subset \mathbb{P}_k^3$$

is smooth over k . For $n \leq 3$ it is rational, for $n = 4$ it is K3 and it is of general type for $n \geq 5$. Shioda and Katsura have shown in [Sh74] and [K-S79] that

Theorem 8.3 (Katsura–Shioda). *The Fermat surface S_n , $n \geq 4$ is unirational in characteristic p if and only if there exists a $\nu \in \mathbb{N}$ such that $p^\nu \equiv -1 \pmod{n}$.*

Shioda [Sh86] has generalized these results to *Delsarte surfaces*. The example of Fermat surfaces shows that being unirational is quite subtle. Namely, one can show (see at the end of this section) that the generic hypersurface of degree $n \geq 4$ in \mathbb{P}_k^3 is *not* unirational, and thus being unirational is not a deformation invariant.

From the point of view of Mori theory it is interesting to note that unirational surfaces that are not rational are covered by *singular* rational curves. However, (unlike in characteristic zero) it is not possible to smoothen these families – after all, possessing a pencil of smooth rational curves implies that the surface in question is rational.

Fundamental group. There do exist geometric obstructions to prevent a surface from being unirational: being dominated by a rational surface, the Albanese morphism of a unirational surface is trivial, and we conclude $b_1 = 0$. Moreover, Serre [Se59] has shown that the fundamental group of a unirational surface is finite, and Crew [Cr84] that it does *not* contain p -torsion in characteristic p . A far more subtle invariant is the formal Brauer group (see below) that can prevent a surface from being unirational (and that may actually be the only obstruction for “nice” classes of surfaces).

K3 surfaces and Shioda-supersingularity. We recall that the *Kummer surface* $\text{Km}(A)$ of an Abelian surface A is the minimal desingularization of the quotient of A by the sign involution. In characteristic $p \neq 2$ the Kummer surface is always a K3 surface. Their unirationality has been settled completely in [Sh77] – in particular, this result shows that there exist unirational K3 surfaces in every positive characteristic $p \geq 3$:

Theorem 8.4 (Shioda). *Let A be an Abelian surface in characteristic $p \geq 3$. Then the Kummer surface $\text{Km}(A)$ is unirational if and only if A is a supersingular as an Abelian variety.*

We recall from Section 6 that an Abelian variety is called supersingular if its p -torsion subgroup scheme $A[p]$ is infinitesimal.

To explain the notion of supersingularity introduced by Shioda [Sh74] let us recall from Section 2 that Igusa's inequality states $\rho \leq b_2$, where ρ denotes the rank of the Néron–Severi group and b_2 is the second Betti number.

Definition 8.5. A surface S is called *supersingular in the sense of Shioda* if $\rho(S) = b_2(S)$ holds true.

This notion is motivated by the following result, also from [Sh74]

Theorem 8.6 (Shioda). *Unirational surfaces are Shioda-supersingular.*

The unirationality results on Kummer and Fermat surfaces show that such a surface is unirational if and only if it is supersingular in the sense of Shioda. These examples thus support the following conjecture:

Conjecture 8.7 (Shioda). A K3 surface is unirational if and only if it is Shioda-supersingular.

Apart from the examples already mentioned this conjecture is known to be true in characteristic 2: the Néron–Severi lattices of Shioda-supersingular K3 surfaces have been classified in [R-S78] and using these results they show (loc. cit.)

Theorem 8.8 (Rudakov–Šafarevič). *Every K3 surface in characteristic 2 that is Shioda-supersingular possesses a quasi-elliptic fibration. In particular, these surfaces are Zariski surfaces and unirational.*

K3 surfaces and Artin-supersingularity. There exists yet another notion of supersingularity, due to Artin [Ar74a]. Namely, for a K3 surface S he considers the functor that associates to every Artin-algebra A the Abelian group

$$\mathrm{Br} : A \mapsto \ker \left(H^2(S \times A, \mathcal{O}_{S \times A}^*) \rightarrow H^2(S, \mathcal{O}_S^*) \right)$$

This functor is pro-representable by a one-dimensional formal group law, the so-called *formal Brauer group* $\widehat{\mathrm{Br}}(S)$ of S .

Now, to every formal group law over a field of positive characteristic, one can assign a *height*, which is a positive integer or infinity. This height measures the complexity of multiplication by p in this formal group, and height infinity means that multiplication by p is equal to zero. Moreover, by a result of Lazard, the height determines the formal group law if the ground field is algebraically closed.

In the case of K3 surfaces the height h of the formal Brauer group satisfies $1 \leq h \leq 10$ or $h = \infty$. Moreover, the height h stratifies the moduli space of K3 surfaces: the generic K3 surface has $h = 1$ – these surfaces are called *ordinary* – and surfaces with $h \geq h_0 + 1$ form a codimension one stratum inside surfaces with $h \geq h_0$ for every h_0 . It remains a 9-dimensional space of K3 surfaces with $h = \infty$.

Definition 8.9. A K3 surface S is called *supersingular in the sense of Artin*, if its formal Brauer group has infinite height.

Shioda-supersingular K3 surfaces are Artin-supersingular and the converse is known to hold for elliptic K3 surfaces [Ar74a]. It is conjectured that all Artin-supersingular K3 surfaces are Shioda-supersingular, that all these surfaces possess elliptic fibrations and are unirational.

To stratify the moduli space of Artin-supersingular K3 surfaces we consider their Néron–Severi groups. Conjecturally, Artin-supersingular K3 surfaces are Shioda-supersingular and thus one expects $\rho = b_2$. In any case, the discriminant of the intersection form on $\text{NS}(S)$ of an Artin-supersingular K3 surface S is equal to

$$\text{disc NS}(S) = \pm p^{2\sigma_0}$$

for some integer $1 \leq \sigma_0 \leq 10$ by [Ar74a].

Definition 8.10. The integer σ_0 is called the *Artin invariant* of the Artin-supersingular K3 surface.

In characteristic $p \geq 3$, Artin-supersingular K3 surfaces with $\sigma_0 \leq 2$ are Kummer surfaces of supersingular Abelian surfaces. Their moduli space is one-dimensional but non-separated. Moreover, there is precisely one such surface with $\sigma_0 = 1$, and it arises as $\text{Km}(E \times E)$, where E is a supersingular elliptic curve. We refer to [Sh79] and [Og79] for details. From the unirationality results on Kummer surfaces in characteristic $p \geq 3$ mentioned above it follows that supersingular K3 surfaces with $\sigma_0 \leq 2$ are unirational. We refer to [Sch05] for the analog results in characteristic 2.

A Torelli theorem for Artin-supersingular K3 surfaces in terms of crystalline cohomology has been shown by Ogus [Og79].

Finally, we refer to [R-S81] for a detailed overview on the geometry of K3 surfaces in positive characteristic.

General type. Let us recall from Section 7 that a minimal surface of general type is called a *Horikawa surface* if it satisfies $K^2 = 2p_g - 4$. This unbounded class is particularly easy to handle since all of them arise as double covers of rational surfaces. In [L-S09] we have constructed unirational Horikawa surfaces in arbitrarily large characteristics and for arbitrarily large p_g . Thus, although the generic Horikawa surface is not unirational, being unirational is nevertheless a very common phenomenon.

Zeta functions. If $k = \mathbb{F}_q$ is a finite field and S is a smooth surface over k then one can count the number $\#S(\mathbb{F}_{q^n})$ of \mathbb{F}_{q^n} -rational points of S and form the *zeta function* of S :

$$Z(S, t) := \exp \left(\sum_{n=1}^{\infty} \#S(\mathbb{F}_{q^n}) \frac{t^n}{n} \right)$$

Then Weil conjectured many properties of $Z(S, t)$, which have been proved using Grothendieck’s theory of étale cohomology. In fact, these conjectures were the historical origin of étale cohomology. We refer to [Hart, Appendix C] or [Mil] for details. In particular, the zeta function of a surface has the form

$$Z(S, t) = \frac{P_1(S, t) \cdot P_3(S, t)}{(1 - qt) \cdot P_2(S, t) \cdot (1 - q^2t)},$$

where $P_j(S, t)$ is a polynomial of degree equal to the j .th Betti number $b_j(S)$. Over the complex numbers $P_2(S, t)$ factors as

$$P_2(S, t) = \prod_{i=1}^{b_2(S)} (1 - \alpha_i t)$$

where the α_i are complex numbers of absolute value q .

Now, suppose S is Shioda-supersingular. Then, after passing to a finite extension, we may assume that all divisor classes of $\text{NS}(S)$ are defined over \mathbb{F}_q . In that case $\alpha_i = \pm q$ holds true, and the zeta function looks like the zeta function of a birationally ruled surface. This perfectly fits into Shioda's conjecture that Shioda-supersingularity for K3 surfaces is related to unirationality. Thus, one might expect that if $\alpha_i = \pm q$ for all i that S is Shioda-supersingular.

More generally, Artin and Tate have conjectured that $P_2(S, t)$ for an arbitrary surface decodes more than just the rank $\rho(S)$ of the Néron–Severi group: namely, let D_1, \dots, D_ρ be independent in $\text{NS}(S)$ and set $B := \sum_i \mathbb{Z}D_i$. Let $\#\text{Br}(S)$ be the order of the Brauer group, which is conjecturally finite. Then

Conjecture 8.11 (Artin–Tate). We get

$$P_2(S, q^{-s}) \sim (-1)^{\rho(S)-1} \cdot \frac{\#\text{Br}(S) \cdot \det(\{D_i \cdot D_j\}_{i,j})}{q^{\chi(\mathcal{O}_S)-1+b_1(S)} \cdot (\text{NS}(S) : B)^2} \cdot (1 - q^{1-s})^{\rho(S)}$$

as s tends to 1

This conjecture has been reduced to a conjecture of Tate [Mil75], and this latter conjecture is known to be true for K3 surfaces, whose formal Brauer groups have finite height [N-O85], as well as for elliptic K3 surfaces [A-S73].

For a relation of these things with Igusa's inequality and a conjecture of Artin and Mazur on Frobenius eigenvalues on crystalline cohomology we refer to [III79, Remarque II.5.13].

Coming back to much more elementary things, we see that the zeta function of a unirational or a Shioda-supersingular surface has a very specific form. For the Fermat surfaces S_n the zeta functions were classically known thanks to results on Gauß- and Jacobi-sums. For example, [K-S79] conclude from these results that S_n is *not* supersingular if there exist no ν such that $p^\nu \equiv -1 \pmod n$.

9. Witt vectors and lifting

This section circles around lifting to characteristic zero. The nicest liftings are those over the Witt ring, and although such liftings exist for curves, it is in general false from dimension two on.

Witt vectors. Let k be a field of positive characteristic p and assume that k is perfect, e.g., algebraically closed or a finite field.

Then one can ask whether there exist rings of characteristic zero having k as residue field. It turns out that there exists a particularly nice ring $W(k)$, the so-called *Witt ring* of k , or *ring of Witt vectors*, which has the following properties:

- (1) $W(k)$ is a discrete valuation ring of characteristic zero,

- (2) the unique maximal ideal \mathfrak{m} of $W(k)$ is generated by p and the residue field R/\mathfrak{m} is isomorphic to k ,
- (3) $W(k)$ is complete with respect to the \mathfrak{m} -adic topology,
- (4) the Frobenius map on k lifts to an additive (but usually not multiplicative) map on $W(k)$,
- (5) there exists another additive map $V : W(k) \rightarrow W(k)$, called *Verschiebung* (German for "shift"), which is zero on the residue field k and such that multiplication by p on $W(k)$ factors as $p = F \circ V = V \circ F$, and finally
- (6) every complete discrete valuation ring with quotient field of characteristic zero and residue field k contains $W(k)$ as subring.

Note that the last property determines $W(k)$ uniquely.

To obtain the Witt ring, one constructs successively rings $W_n(k)$ starting from $W_1(k) := k$ and then by definition $W(k)$ is the projective limit over all these $W_n(k)$, cf. [Se68, Chapitre II.6]. The main example to bear in mind is the following:

Example 9.1. For the finite field \mathbb{F}_p we have $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$ and thus

$$W(\mathbb{F}_p) = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

is isomorphic to \mathbb{Z}_p , the ring of p -adic integers. One sees directly that the maximal ideal of $W(\mathbb{F}_p)$ is generated by p and that $W(\mathbb{F}_p)$ is complete with respect to the p -adic topology. In this case, F is the identity on $W(\mathbb{F}_p)$ and V is multiplication by p .

Lifting over the Witt ring. Let X be a scheme of finite type over some field k of positive characteristic p . Then there are different notions of what it means to lift X to characteristic zero. To be precise, let R be a ring of characteristic zero with maximal ideal \mathfrak{m} , having residue field $R/\mathfrak{m} \cong k$. For example, we could have $R = W(k)$ and $\mathfrak{m} = (p)$.

Definition 9.2. A *lifting* (resp. *formal lifting*) of X over R is a scheme (resp. formal scheme) \mathcal{X} of finite type and flat over $\mathrm{Spec} R$ (resp. $\mathrm{Spf} R$) with special fiber X .

In case $R = W(k)$, i.e., if X admits a lifting over the Witt ring, then many "characteristic p pathologies" cannot happen. We have already encountered the following results in Section 2:

- (1) if X is of dimension $d \leq p$ and lifts over $W_2(k)$ then its Frölicher spectral sequence from Hodge to deRham-cohomology degenerates at E_1 by a result of Deligne and Illusie, see [D-I87] and [III02, Corollary 5.6],
- (2) if X is of dimension $d \leq p$ and lifts over $W_2(k)$, then ample line bundles satisfy Kodaira vanishing, see [D-I87] and [III02, Theorem 5.8],
- (3) if X lifts over $W(k)$ then crystalline cohomology coincides with the deRham-cohomology of $\mathcal{X}/W(k)$.

Actually, the last property was historically the starting point of crystalline cohomology: Grothendieck realized that if a smooth variety lifts over $W(k)$ that then the deRham-cohomology of the lift does not depend on the chosen lift and one gets

a well-defined cohomology theory with values in $W(k)$. The main technical point to overcome defining crystalline cohomology for arbitrary smooth varieties is that they usually do *not* lift over $W(k)$. We refer to [B-O] for details.

Example 9.3. Smooth curves and birationally ruled surfaces lift over the Witt ring by Grothendieck’s existence theorem [III05, Theorem 5.19].

Lifting over more general rings. Let R be an integral ring with maximal ideal \mathfrak{m} , residue field $R/\mathfrak{m} \cong k$ and quotient field K of characteristic zero. Let X be a smooth projective variety over k , let \mathcal{X} be a lifting of X over $\text{Spec } R$ and denote by $\mathcal{X}_K \rightarrow \text{Spec } K$ its generic fiber.

Choosing a DVR dominating (R, \mathfrak{m}) and passing to the \mathfrak{m} -adic completion we may always assume that (R, \mathfrak{m}) is a local DVR that is \mathfrak{m} -adically complete. By the universal property of the Witt ring, R contains $W(k)$ and \mathfrak{m} lies above $(p) \subset W(k)$. Then it makes sense to talk about the *ramification index* of R over $W(k)$.

To give a flavor of the subtleties that occur let us mention the following

- (1) Abelian varieties admit a formal lifting over the Witt ring by an unpublished result of Grothendieck. However, to get an algebraic lifting one would like to have an ample line bundle on the formal lifting in order to apply Grothendieck’s existence theorem, see [III05, Theorem 4.10]. However, even if one succeeds in doing so this is usually at the prize that this new formal lifting (which then is algebraic) may exist over a ramified extension of the Witt ring only. For Abelian varieties this has been established by Mumford.
- (2) K3 surfaces have no global vector fields by a result of Rudakov and Šafarevič [R-S76], which implies $H^2(X, \Theta_X) = 0$ and thus K3 surfaces admit formal liftings over the Witt ring. Deligne [Del81] has shown that one can lift with every K3 surface also an ample line bundle, which gives an algebraic lifting - again at the prize that this lifting may exist over ramified extensions of the Witt ring only.
- (3) it is known (unpublished result by Ekedahl and Shepherd-Barron) that Enriques surfaces - even in characteristic 2 - lift to characteristic zero. However, their Frölicher spectral sequences may not degenerate at E_1 . Thus, these latter surfaces only lift over *ramified* extensions of the Witt ring, but not over the Witt ring itself.

However, even if X lifts “only” over a ramified extension of the Witt ring this does imply something: Flatness of \mathcal{X} over $\text{Spec } R$ implies that $\chi(\mathcal{O})$ of special and generic fiber coincide and smoothness of \mathcal{X} over $\text{Spec } R$ implies that the étale Betti numbers of special and generic fiber coincide. For surfaces there are further results due to Katsura and Ueno, see [K-U85, Section 9]. Since we are mainly interested in surfaces let us summarize these results:

Theorem 9.4. *Let \mathcal{S} be a lifting of the smooth projective surface S over $\text{Spec } R$ with generic fiber \mathcal{S}_K . Then,*

$$\begin{aligned} b_i(\mathcal{S}) &= b_i(\mathcal{S}_K) & c_2(\mathcal{S}) &= c_2(\mathcal{S}_K) \\ \chi(\mathcal{O}_{\mathcal{S}}) &= \chi(\mathcal{O}_{\mathcal{S}_K}) & K_{\mathcal{S}}^2 &= K_{\mathcal{S}_K}^2 \\ \kappa(\mathcal{S}) &= \kappa(\mathcal{S}_K) \end{aligned}$$

Moreover, \mathcal{S} is minimal if and only \mathcal{S}_K is minimal.

If S is of general type then $P_n(S) = P_n(\mathcal{S}_K)$ for $n \geq 3$ since these numbers depend on χ and K^2 by Riemann–Roch and [Ek88, Theorem II.1.7]. However, in general, $p_g(S)$ may differ from $p_g(\mathcal{S}_K)$, as the examples in [Se58] show. More precisely, Hodge invariants are semi-continuous, i.e., in general we have

$$h^{i,j}(\mathcal{S}) \geq h^{i,j}(\mathcal{S}_K) \quad \text{for all } i, j \geq 0.$$

In case of equality for all i, j the Frölicher spectral sequence of S degenerates at E_1 . These results imply that there exist smooth projective varieties from dimension two on that do not admit any sort of lifting, namely:

Examples 9.5. Let S be

- (1) a minimal surface of general type with $K_S^2 > 9\chi(\mathcal{O}_S)$, i.e., violating the Bogomolov–Miyaoka–Yau inequality, or
- (2) quasi-elliptic surface with $\kappa(S) = 1$ and $\chi(\mathcal{O}_S) < 0$.

Then S does not admit a lifting whatsoever, not even over ramified extensions of the Witt ring.

For this and related questions, see also the discussion in [III05, Section 5F].

10. Singularities

Resolution of singularities over algebraically closed fields of positive characteristic is a difficult topic and still not known to hold.

For curves the situation is simple: normalization gives the desired resolution and likewise successive blow-ups in closed points eventually resolve singularities.

Resolution of surface singularities is already more involved but has been established by Abhyankar. Later work of Artin shows that the theory of rational singularities and Du Val singularities in positive characteristic is very similar to characteristic zero.

It is also known, again by work of Abhyankar, that 3-fold singularities can be resolved in characteristic $p \geq 7$. In higher dimensions, it is known that varieties are generically finitely dominated by smooth varieties, by de Jong’s theory of alterations. Finally, from dimension three on, it is not so clear what the “right” definitions of rational or canonical singularities will be.

Resolution of singularities. Let us drop the assumption on projectivity and smoothness for this section and let S be an integral two-dimensional scheme of finite type over an algebraically closed field k .

Passing to the normalization, we may assume that S is normal in which case the non-smooth locus of S consists of isolated points. Then, *resolution of surface*

singularities over algebraically closed fields of arbitrary characteristic has been proved by Abhyankar [Ab56]:

Theorem 10.1 (Abhyankar). *For normal S there exists a proper morphism*

$$\varphi : \tilde{S} \rightarrow S$$

where \tilde{S} is smooth over k , and φ is an isomorphism outside the non-smooth locus of S and whose fibers are one-dimensional.

Moreover, we may assume that the fibers are normal-crossing divisors and after contracting rational (-1) -curves in the fibers one may assume that the resolution is *minimal*, i.e., does not contain any exceptional (-1) -curves.

Artin [Ar74b] proved simultaneous resolution for surface singularities in families - however at the price that this resolution generally exists only in the category of algebraic spaces, which is slightly more general than the category of schemes. Note that apart from resolution of singularities for 3-folds in characteristic $p \geq 7$, also due to Abhyankar, no more general results are available in positive characteristic and in higher dimensions.

Now, assume that $\varphi : \tilde{S} \rightarrow S$ is a minimal resolution of singularities and assume that there is precisely one singular point $P \in S$. Consider the scheme-theoretic fiber

$$Z := \varphi^{-1}(\mathfrak{m}_P) \cdot \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{\tilde{S}} \left(\sum_i r_i E_i \right)$$

and assume that φ is chosen s.th. this fiber is a normal crossing divisor with smooth and irreducible curves E_i . Then the intersection matrix $(E_i \cdot E_j)_{ij}$ is symmetric and negative definite. Conversely, given a normal crossing divisor $Z = \bigcup_i Z_i$ on a smooth surface S such that the corresponding intersection matrix is negative definite then there exists a morphism $S \rightarrow Y$ contracting Z . However, usually Y is not a scheme but only an algebraic space [Ar70]. We refer to [Ba, Chapter 3] for examples and discussion.

Rational singularities. Let $\varphi : \tilde{S} \rightarrow S$ be a minimal resolution of singularities of a normal surface singularity $P \in S$ with scheme-theoretic fiber $Z = \sum_i r_i Z_i$ as above.

Proposition 10.2 (Artin). *The following are equivalent:*

- (1) $R^1 \varphi_* \mathcal{O}_{\tilde{S}} = 0$
- (2) $\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_S)$
- (3) $h^1(Z', \mathcal{O}_{Z'}) = 0$ for every divisor $Z' > 0$ with support contained in Z

Definition 10.3. If these equivalent conditions hold then the singularity $P \in S$ is called *rational*.

The following theorem shows that the problem that the contraction morphism may exist only in the category of algebraic spaces does not happen for rational singularities.

Theorem 10.4 (Artin). *Let S be a smooth projective surface and E a connected curve with integral components E_i . Then the following are equivalent*

- (1) *there exists a morphism $\varphi : S \rightarrow Y$ such that Y is a normal projective surface, $\varphi(E)$ is a point $y \in Y$, φ is an isomorphism from $S - E$ to $Y - \{y\}$, and $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_Y)$*
- (2) *the intersection matrix $(E_i \cdot E_j)_{ij}$ is negative definite and $h^1(Z', \mathcal{O}_{Z'}) = 0$ for every positive divisor Z' contained in E .*

We refer to [Ar62], [Ar66] and [Ba, Chapter 3] for details and further references, where fundamental cycles and embedding dimensions of rational singularities are discussed.

We remark that if a resolution $\varphi : \tilde{S} \rightarrow S$ of a normal surface singularity $P \in S$ satisfies $R^1\varphi_*\mathcal{O}_{\tilde{S}} = 0$ (which implies that all resolutions have this property) that then $R^1\varphi_*\omega_{\tilde{S}} = 0$ automatically holds true. In higher dimensions and in characteristic zero, $R^i\varphi_*\mathcal{O}_{\tilde{S}} = 0$ for all $i \geq 1$ implies $R^i\varphi_*\omega_{\tilde{S}} = 0$ for all $i \geq 1$ by the Grauert–Riemenschneider vanishing theorem. Since this theorem may fail in positive characteristic, the vanishing of $R^i\varphi_*\omega_{\tilde{S}}$ has to be postulated explicitly for rational singularities in higher dimensions. However, see [B-K05, Chapter 1] for this vanishing result for F -split singularities.

Rational Gorenstein singularities. Among the rational singularities those that are Gorenstein, i.e., where the dualizing sheaf is locally free, are characterized as follows

Theorem 10.5 (Artin). *Let S be a smooth projective surface and E a connected curve with integral components E_i . Then the following are equivalent*

- (1) *there exists a morphism $\varphi : S \rightarrow Y$ such that Y is a normal projective surface, $\varphi(E)$ is a Gorenstein point $y \in Y$, φ is an isomorphism from $S - E$ to $Y - \{y\}$, and $\varphi^*\omega_Y = \omega_S$.*
- (2) *the intersection matrix $(E_i \cdot E_j)_{ij}$ is negative definite, the E_i are smooth rational curves and $E_i^2 = -2$ for all i .*

Definition 10.6. The rational Gorenstein surface singularities are also called *Rational double points* or *DuVal singularities*.

As in characteristic zero, the *dual resolution graph* of a rational Gorenstein surface singularity is a Dynkin diagram of type A_n , D_n , E_6 , E_7 or E_8 .

Since the embedding dimension of these singularities is three, all of them are hypersurface singularities, and explicit equations in all characteristics have been worked out in [Ar77]. As a result of this explicit classification, Artin obtains

Theorem 10.7. *To every Dynkin diagram Γ there exist only a finite number of analytic isomorphism classes of rational Gorenstein singularities with dual resolution graph Γ . If $p \geq 7$ then D determines the singularity up to analytic isomorphism.*

On the other hand, there are for example five different analytic isomorphism classes of E_8 -singularities in characteristic 2.

In [Ar77], also the local fundamental groups of these singularities are computed. For example, the A_{p-1} -singularity has trivial local fundamental group in characteristic p . Thus, these singularities are not characterized as quotients of smooth surfaces by finite subgroups of SL_2 . On the other hand, A_{p-1} is characterized as quotient of a smooth surfaces by the finite and infinitesimal group scheme μ_p , see [Hir99b], [R-S76] and Section 11.

11. Inseparable morphisms and foliations

In this section we study inseparable morphisms of height one in greater detail. On the level of function fields this is Jacobson's correspondence, a kind of Galois correspondence for purely inseparable field extensions. However, this correspondence is not via automorphisms but via derivations. On the level of geometry this translates into foliations in tangent sheaves. In the case of surfaces this simplifies to p -closed vector fields.

Jacobson's correspondence. Let us recall the Galois correspondence: given a field K and a *separable* extension L there exists a minimal Galois extension of K containing L , the Galois closure of L . Moreover, L/K is a finite Galois extension the Galois group G is defined to be the automorphism group of L over K , which is finite of degree equal to the index $[L : K]$. Finally, there is a bijective correspondence between subgroups of G and intermediate fields $K \subseteq M \subseteq L$. In particular, there are only finitely many fields between K and L .

In Section 1 we encountered height 1-extensions of a field K . It turns out that automorphism of purely inseparable extensions are trivial, and thus give no insight into these extensions. However, there does exist a sort Galois-type correspondence for such extensions, *Jacobson's correspondence* [Jac64, Chapter IV]. Instead of automorphisms one studies *derivations* over K .

Namely, let L be a purely inseparable extension of height 1 of K , i.e., $K \subseteq L \subseteq K^{p^{-1}}$, or, put differently, $L^p \subseteq K$. Consider the Abelian group

$$\mathrm{Der}(L) := \{ \delta : K^{p^{-1}} \rightarrow K^{p^{-1}}, \delta \text{ is a derivation and } \delta(L) = 0 \}.$$

Since $\delta(x^p) = p \cdot x^{p-1} \cdot \delta(x) = 0$, these derivations are automatically K -linear and thus $\mathrm{Der}(L)$ is a K -vector space. More precisely, $\mathrm{Der}(L)$ is a subvector space of $\mathrm{Der}(K)$ and this latter is n -dimensional, where $n = \mathrm{tr.deg}_k K$.

But this vector space carries more structure: usually, if δ and η are derivations, then their composition $\delta \circ \eta$ is no derivation, which is why one studies their *Lie bracket* instead, i.e., the commutator $[\delta, \eta] = \delta \circ \eta - \eta \circ \delta$, which is a derivation. Now, over fields of positive characteristic p it turns out that the p -fold composite $\delta \circ \dots \circ \delta$ is a derivation. The reason is, that expanding this composition the binomial coefficients occurring that usually prevent this composition from being a derivation are all divisible by p , i.e., vanish. This p -power operation is denoted by $\delta \mapsto \delta^{[p]}$. The K -vector space $\mathrm{Der}(L)$ is thus closed under the Lie bracket and the p -power operation.

Definition 11.1. A *p-Lie algebra* or *restricted Lie algebra* is a Lie algebra over a field of characteristic p together with a p -power map $\delta \mapsto \delta^{[p]}$ satisfying the axioms in [Jac62, Chapter V.7].

By a remarkable result of Jacobson the p -power map already determines the Lie bracket uniquely albeit in a way that is not so easy to write down explicitly. We will thus omit the Lie bracket from our considerations.

So far we have associated to every finite and purely inseparable extension L/K of height one a sub- p -Lie algebra of $\text{Der}(K)$. Conversely, given such a Lie algebra $(V, -^{[p]})$ we may form the fixed set

$$(K^{p^{-1}})^{(V, -^{[p]})} := \{x \in K^{p^{-1}} \mid \delta(x) = 0 \ \forall \delta \in V\},$$

which is easily seen to be a field. Since elements of V are K -linear derivations this fixed field contains K . Moreover, by construction it is contained in $K^{p^{-1}}$, i.e., of height one.

Theorem 11.2 (Jacobson). *There is a bijective correspondence*

$$\{\text{height one extensions of } K\} \leftrightarrow \{\text{sub-}p\text{-Lie algebras of } \text{Der}(K)\}$$

Let us mention one important difference to Galois theory: suppose K is of transcendence degree n over a perfect field k , which we think of as the function field of an n -dimensional variety over k . Then the extension $K^{p^{-1}}/K$ is finite of degree p^n . But for $n \geq 2$ there are usually infinitely many sub- p -Lie algebras of $\text{Der}(K)$ and thus infinitely many fields between K and $K^{p^{-1}}$.

Curves. Let C be a smooth projective curve over a perfect field k with function field $K = k(C)$. Then the field extension $K^p \subset K$ corresponds to the k -linear Frobenius morphism $F : C \rightarrow C^{(p)}$ and is an extension of degree p .

Since every purely inseparable extension L/K of degree p is of the form $L = K[\sqrt[p]{x}]$ for some $x \in K$, such extensions are of height one, i.e., $K \subseteq L \subseteq K^{p^{-1}}$. Simply for degree reasons this implies that the only purely inseparable morphism of degree p between normal curves is the k -linear Frobenius morphism. Since every finite purely inseparable field extension can be factored successively into extensions of degree p we see

Theorem 11.3. *Let C and D be normal curves over a perfect field k and let $\varphi : C \rightarrow D$ be a purely inseparable morphism of degree p^n . Then φ is the n -fold composite of the k -linear Frobenius morphism.*

Foliations. From dimension two on there are many more purely inseparable morphisms than just compositions of Frobenius. In fact if X is an n -dimensional variety over a perfect field with $n \geq 2$ then the k -linear Frobenius morphism has degree p^n and it factors over many height one morphisms.

To classify height one morphisms $\varphi : X \rightarrow Y$ from a fixed smooth variety X over a perfect field k we geometrize Jacobson's correspondence

Definition 11.4. A *foliation* on a smooth variety X is a saturated subsheaf \mathcal{E} of the tangent sheaf Θ_X that is closed under the p -power operation.

Then Jacobson's correspondence translates into

Theorem 11.5. *There is a bijective correspondence*

$$\left\{ \begin{array}{l} \text{finite morphisms } \varphi : X \rightarrow Y \\ \text{of height one with } Y \text{ normal} \end{array} \right\} \leftrightarrow \{ \text{foliations in } \Theta_X \}$$

The saturation assumption is needed because otherwise a foliation and its saturation define the same extension of function fields, and thus correspond to the same normal variety. We refer to [Ek87] for details.

Surfaces. Thus, in order to describe finite height one morphisms $\varphi : X \rightarrow Y$ from a smooth surface onto a normal surface we have to consider foliations inside Θ_X . The sheaf Θ_X and its zero subsheaf correspond to the k -linear Frobenius morphism and the identity, respectively. Thus, height one-morphisms of degree p correspond to foliations of rank one in Θ_X .

To simplify our exposition consider \mathbb{A}_k^2 , i.e., $X = \text{Spec } R$ with $R = k[x, y]$. Then Θ_X corresponds to the R -module generated by the partial derivatives $\partial/\partial x$ and $\partial/\partial y$. Now, a finite morphism of height one $\varphi : X \rightarrow Y$ with Y normal corresponds to a ring extension

$$R^p = k[x^p, y^p] \subseteq S \subseteq R = k[x, y]$$

where S is normal. By Jacobson's correspondence, giving S is equivalent to giving a foliation inside Θ_X , which will be of rank one if $S \neq R, R^p$. This amounts to giving a regular vector field (actually, it may be that we have choose an open affine cover and regular vector fields on them, but assume we do not have to)

$$\delta = f(x, y) \frac{\partial}{\partial x} + g(x, y) \frac{\partial}{\partial y}$$

for some $f, g \in R$. Being closed under p .th powers translates into

$$\delta^{[p]} = h(x, y) \cdot \delta \quad \text{for some } h(x, y) \in R,$$

i.e., δ is a p -closed vector field.

We may assume that f and g are coprime and then the zero set of the ideal (f, g) is of codimension two and is called the *singular locus* of the vector field. It is not so difficult to see that S is smooth over k outside the singular locus of δ , cf. [R-S76].

Finally, a purely inseparable morphism $\varphi : X \rightarrow Y$ is everywhere ramified, i.e., $\Omega_{X/Y}$ has support on the whole of X . Nevertheless, the canonical divisor classes of X and Y are related by a kind of Riemann–Hurwitz formula and the role of the ramification divisor is played by a divisor class that can be read off from the foliation, see [R-S76].

Quotients by group schemes. Let X be a smooth but not necessarily projective variety of any dimension over a perfect field k . We have seen that a global section

$$\delta \in H^0(X, \Theta_X)$$

gives rise to an inseparable morphism of degree p and height one if and only if δ is p -closed, i.e., $\delta^{[p]} = c \cdot \delta$ for some $c \in H^0(X, \mathcal{O}_X)$.

Definition 11.6. A vector field δ is called *multiplicative* if $\delta^{[p]} = \delta$ and it is called *additive* if $\delta^{[p]} = 0$.

Let δ be additive or multiplicative. Applying a (truncated) exponential series to δ one obtains on X an action of some finite and flat group scheme G , which is infinitesimal and of length p , see [Sch05, Section 1]. Then the inseparable morphism $\varphi : X \rightarrow Y$ corresponding to δ is in fact a quotient morphism $X \rightarrow X/G$. Moreover, the 1-dimensional p -Lie algebra generated by δ is the p -Lie algebra to G . This latter is the Zariski tangent space of G with p .th power map coming from Frobenius. Let us recall from Section 1 that the only infinitesimal group schemes of length p are α_p and μ_p . Finally, one obtains

Theorem 11.7. *Additive (multiplicative) vector fields correspond to inseparable morphisms of degree p that are quotients by the group scheme α_p (resp. μ_p).*

This also explains the name additive (resp. multiplicative), since α_p (resp. μ_p) is a subgroup scheme of the additive group \mathbb{G}_a (resp. multiplicative group \mathbb{G}_m).

Singularities. Let us now assume that X is a smooth surface and let δ be a multiplicative vector field. By a result of [R-S76], the vector field can be written near a singularity in local coordinates x, y as

$$\delta = x \frac{\partial}{\partial x} + a \cdot y \frac{\partial}{\partial y} \quad \text{for some } a \in \mathbb{F}_p^\times$$

Let $\varphi : X \rightarrow Y$ be the inseparable morphism corresponding to δ . In [Hir99b] it has been shown that Y has toric singularities of type $\frac{1}{p}(1, a)$. Thus, quotients by μ_p behave very much as one would expect it from cyclic quotient singularities.

On the other hand, quotients by α_p are much more complicated - the singularities need not even be rational and we refer to [Li08] for examples.

Bibliography

- [Ab56] S. S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$* , Ann. of Math. 63, 491-526 (1956).
- [Ar62] M. Artin, *Some numerical criteria for contractibility of curves on algebraic surfaces*, Amer. J. Math. 84, 485-496 (1962).
- [Ar66] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. 88, 129-136 (1966).
- [Ar70] M. Artin, *Algebraization of formal moduli. II. Existence of modifications*, Ann. of Math. 91, 88-135 (1970).
- [Ar74a] M. Artin, *Supersingular $K3$ surfaces*, Ann. Sci. École Norm. Sup. (4), 543-567 (1974).
- [Ar74b] M. Artin, *Algebraic construction of Brieskorn's resolutions*, J. Algebra 29, 330-348 (1974).
- [Ar77] M. Artin, *Coverings of the rational double points in characteristic p* , in Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 11-22.
- [A-S73] M. Artin, H. P. F. Swinnerton-Dyer, *The Shafarevich-Tate conjecture for pencils of elliptic curves on $K3$ surfaces*, Invent. Math. 20, 249-266 (1973).
- [Ba] L. Badescu, *Algebraic Surfaces*, Springer Universitext 2001.
- [BHH87] G. Barthel, F. Hirzebruch, T. Höfer, *Geradenkongurationen und algebraische Flächen*, Aspekte der Mathematik D4, Vieweg (1987).
- [B-O] P. Berthelot, A. Ogus, *Notes on crystalline cohomology*, Princeton University Press 1978.
- [B-M2] E. Bombieri, D. Mumford, *Enriques classification of surfaces in char. p , II*, in Complex analysis and algebraic geometry, Cambridge Univ. Press, 1977, pp. 2342.
- [B-M3] E. Bombieri, D. Mumford, *Enriques classification of surfaces in char. p , III*, Invent. Math. 35, 197-232 (1976).
- [B-K05] M. Brion, S. Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory*, Progress in Mathematics 231, Birkhäuser 2005.
- [C-D] F. Cossec, I. Dolgachev, *Enriques surfaces. I*, Progress in Mathematics 76, Birkhäuser 1989.
- [Cr84] R. Crew, *Étale p -covers in characteristic p* , Compositio Math. 52, 31-45 (1984).
- [Del81] P. Deligne, *Relèvement des surfaces $K3$ en caractéristique nulle*, Lecture Notes in Math. 868, 58-79 (1981).
- [D-I87] P. Deligne, L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. 89, 247-270 (1987).
- [Ea08] R. Easton, *Surfaces violating Bogomolov-Miyaoka-Yau in positive characteristic*, Proc. Amer. Math. Soc. 136, 2271-2278 (2008).
- [Ek87] T. Ekedahl, *Foliations and inseparable morphisms*, Algebraic geometry Bowdoin 1985, Proc. Symp. Pure Math. 46, Part 2, 139-146 AMS 1987.
- [Ek88] T. Ekedahl, *Canonical models of surfaces of general type in positive characteristic*, Inst. Hautes Études Sci. Publ. Math. No. 67, 97-144 (1988).
- [Go96] D. Goss, *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete 35, Springer 1996.
- [Hart] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer 1977.
- [Hir99a] M. Hirokado, *A non-liftable Calabi-Yau threefold in characteristic 3*, Tohoku Math. J. 51, 479-487 (1999).

- [Hir99b] M. Hirokado, *Singularities of multiplicative p -closed vector fields and global 1-forms of Zariski surfaces*, J. Math. Kyoto Univ. 39, 455-468 (1999).
- [Ig55a] J. Igusa, *A Fundamental Inequality in the Theory of Picard varieties*, Proc. Nat. Acad. Sci. USA 41, 317-320 (1955).
- [Ig55b] J. I. Igusa, *On Some Problems in Abstract Algebraic Geometry*, Proc. Nat. Acad. Sci. USA 41, 964-967 (1955).
- [Ig60] J. I. Igusa, *Betti and Picard numbers of abstract algebraic surfaces*, Proc. Nat. Acad. Sci. USA 46, 724-726 (1960).
- [Ill79] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. scient. Ec. Norm. Sup. 12, 501-661 (1979).
- [Ill02] L. Illusie, *Frobenius and Hodge Degeneration*, in Introduction to Hodge theory, SMF/AMS Texts and Monographs 8. AMS 2002.
- [Ill05] L. Illusie, *Grothendieck's existence theorem in formal geometry*, AMS Math. Surveys Monogr. 123, Fundamental algebraic geometry, 179-233 (2005).
- [Jac62] N. Jacobson, *Lie algebras*, Republication of the 1962 original, Dover Publications 1979.
- [Jac64] N. Jacobson, *Lectures in abstract algebra. Vol III: Theory of fields and Galois theory*, D. Van Nostrand Co. 1964.
- [K-S79] T. Katsura, T. Shioda, *On Fermat varieties*, Tohoku Math. J. 31, 97-115 (1979).
- [K-U85] T. Katsura, K. Ueno, *On elliptic surfaces in characteristic p* , Math. Ann. 272, 291-330 (1985).
- [La79] W. E. Lang, *Quasi-elliptic surfaces in characteristic three*, Ann. Sci. École Norm. Sup. 12, 473-500 12 (1979).
- [La81] W. E. Lang, *Classical Godeaux surface in characteristic p* , Math. Ann. 256, 419-427 (1981).
- [La83a] W. E. Lang, *Examples of surfaces of general type with vector fields*, Arithmetic and geometry Vol. II, 167-173, Progress in Mathematics 36, Birkhäuser 1983.
- [La83b] W. E. Lang, *On Enriques surfaces in characteristic p . I*, Math. Ann. 265, 45-65 (1983).
- [La88] W. E. Lang, *On Enriques surfaces in characteristic p . II*, Math. Ann. 281, 671-685 (1988).
- [Li08] C. Liedtke, *Uniruled surfaces of general type*, Math. Z. 259, 775-797 (2008).
- [Li08] C. Liedtke, *Algebraic Surfaces of General Type with Small c_1^2 in Positive Characteristic*, Nagoya Math. J. 191, 111-134 (2008).
- [Li09a] C. Liedtke, *A note on non-reduced Picard schemes*, J. Pure Appl. Algebra 213, 737-741 (2009).
- [Li09b] C. Liedtke, *Non-classical Godeaux surfaces*, Math. Ann. 343, 623-637 (2009).
- [L-S09] C. Liedtke, M. Schütt, *Unirational Surfaces on the Noether Line*, Pacific J. Math. 239, 343-356 (2009).
- [Liu02] Q. Ling, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics 6, Oxford University Press 2002.
- [Mat80] H. Matsumura, *Commutative Algebra*, second edition, Mathematics Lecture Note Series 56. Benjamin/Cummings Publishing 1980.
- [Mil75] J. Milne, *On a conjecture of Artin and Tate*, Ann. of Math. 102, 517-533 (1975).
- [Mil] J. Milne, *Étale cohomology*, Princeton Mathematical Series 33, Princeton University Press 1980.
- [Mir84] R. Miranda, *Nonclassical Godeaux surfaces in characteristic five*, Proc. Amer. Math. Soc. 91, 9-11 (1984).
- [M-S03] S. Mori, N. Saito *Fano threefolds with wild conic bundle structures*, Proc. Japan Acad. Ser. A Math. Sci. 79, 111-114 (2003).
- [Mu61] D. Mumford, *Pathologies of modular algebraic surfaces*, Amer. J. Math. 83, 339-342 (1961).
- [Mu66] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies 59, Princeton University Press 1966.

- [Mu69] D. Mumford, *Enriques' classification of surfaces in char.p. I* in Global Analysis (Papers in Honor of K. Kodaira) 1969, pp. 325-339.
- [Mu70] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics 5, Oxford University Press 1970.
- [N-O85] N. Nygaard, A. Ogus, *Tate's conjecture for K3 surfaces of finite height*, Ann. of Math. 122, 461-507 (1985).
- [Og79] A. Ogus, *Supersingular K3 crystals*, Journées de Géométrie Algébrique de Rennes Vol. II, Astérisque 64, 3-86 (1979).
- [O-T70] F. Oort, J. Tate, *Group schemes of prime order* Ann. Sci. cole Norm. Sup. 3, 1-21 (1970).
- [Ra78] M. Raynaud, *Contre-exemple au "vanishing theorem" en caractéristique $p > 0$* , in C.P. Ramanujam A tribute, Springer 1978, pp. 273-278.
- [R-S76] A. N. Rudakov, I. R. Šafarevič, *Inseparable morphisms of algebraic surfaces*, Izv. Akad. Nauk SSSR 40, 1269-1307 (1976).
- [R-S78] A. N. Rudakov, I. R. Šafarevič, *Supersingular K3 surfaces over fields of characteristic 2*, Izv. Akad. Nauk SSSR 42, 848-869 (1978).
- [R-S81] A. N. Rudakov, I. R. Šafarevič, *Surfaces of type K3 over fields of finite characteristic*, Current problems in mathematics Vol. 18, Akad. Nauk SSSR, 115-207 (1981)
- [Sch04] S. Schröer, *Some Calabi-Yau threefolds with obstructed deformations over the Witt vectors*, Compos. Math. 140, 1579-1592 (2004).
- [Sch05] S. Schröer, *Kummer surfaces for the self-product of the cuspidal rational curve*, J. Algebraic Geom. 16, 305-346 (2007).
- [Sch09] S. Schröer, *On genus change in algebraic curves over imperfect fields*, Proc. AMS 137, 1239-1243 (2009).
- [Se58] J. P. Serre, *Sur la topologie des variétés algébriques en caractéristique p* , Symposium internacional de topologia algebraica, 24-53, Mexico City, 1958.
- [Se59] J. P. Serre, *On the fundamental group of a unirational variety*, J. London Math. Soc. 34, 481-484 (1959).
- [Se68] J. P. Serre, *Corps locaux*, 3ème edition, Publications de l'Université de Nancago VIII, Hermann, 1968
- [SB91a] N. I. Shepherd-Barron, *Unstable vector bundles and linear systems on surfaces in characteristic p* , Invent. math. 106, 243-262 (1991).
- [SB91b] N. I. Shepherd-Barron, *Geography for surfaces of general type in positive characteristic*, Invent. Math. 106, 263-274 (1991).
- [Sh74] T. Shioda, *An example of unirational surfaces in characteristic p* , Math. Ann. 211, 233-236 (1974).
- [Sh77] T. Shioda, *Some results on unirationality of algebraic surfaces*, Math. Ann. 230, 153-168 (1977).
- [Sh79] T. Shioda, *Supersingular K3 surfaces*, Lecture Notes in Math. 732, Springer, 564-591 (1979).
- [Sh86] T. Shioda, *An explicit algorithm for computing the Picard number of certain algebraic surfaces*, Amer. J. Math. 108, 415-432 (1986).
- [Sil86] J. H. Silverman, *The Arithmetic of Elliptic Curves*, GTM 106, Springer 1986
- [Sil94] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, GTM 151, Springer 1994
- [Sz79] L. Szpiro, *Sur le théorème de rigidité de Parsin et Arakelov*, Astérisque 64, Soc. Math. France 1979.
- [Ta52] J. Tate, *Genus change in inseparable extensions of function fields*, Proc. AMS 3, 400-406 (1952).
- [Wa79] W. C. Waterhouse, *Introduction to Affine Group schemes*, GTM 66, Springer 1979.
- [Za58] O. Zariski, *On Castelnuovo's criterion of rationality $p_a = P_2 = 0$ of an algebraic surface*, Illinois J. Math. 2, 303-315 (1958).

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